Inscribed quadrilateral in a circle

See Fig. 1. D is the center of the circle, and A, B, and C are on the circle. If $\angle ADC$ is θ, what is $\angle ABC$ in terms of θ?

See Fig. 2. Let's say

$$\angle ABD = a, \qquad \angle CBD = b \tag{1}$$

Then, we have

$$\angle ABC = a + b \tag{2}$$

As $\triangle ABD$ and $\triangle CBD$ are isosceles triangles, we have

$$\angle DAB = a, \qquad \angle DCB = b \tag{3}$$

As the sum of angles of a triangle is always $180°$, we have

$$\angle ADB = 180° - 2a, \qquad \angle BDC = 180° - 2b \tag{4}$$

which implies

$$\angle ADE = 2a, \qquad \angle CDE = 2b \tag{5}$$

which, in turn, implies

$$\angle ADC = 2a + 2b = 2(a + b) \tag{6}$$

As we defined $\angle ADC = \theta$, from (2) we see that

$$\angle ABC = \theta/2 \tag{7}$$

Notice that once the positions A and C on a circle are determined, the angle θ is determined.

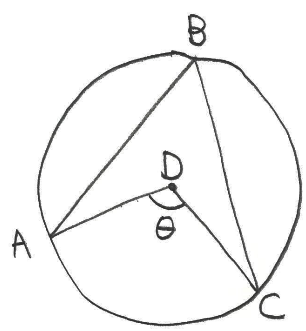

Figure 1: $\angle ADC = \theta$, $\angle ABC =?$

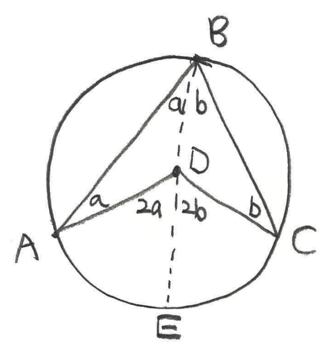

Figure 2: $\angle ABC = a + b$, $\angle ADC = 2a + 2b$

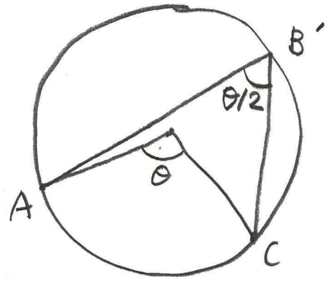

Figure 3: $\angle AB'C = \theta/2$

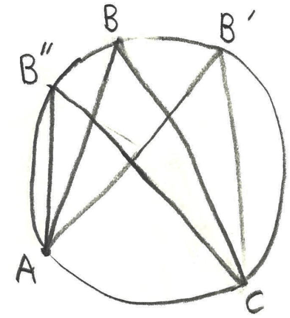

Figure 4: $\angle ABC = \angle AB'C = \angle AB''C$

$\angle ABC$, which is $\theta/2$, is also determined, no matter where B is located, as long as it is on the circle. Thus, for example, as in Fig. 3. even if B were located at B', $\angle AB'C$ is still $\theta/2$. Think along this way. We could as well not draw $\angle ADC$ from the first place, as in Fig. 4. Then, we see that

$$\angle ABC = \angle AB'C = \angle AB''C \tag{8}$$

i.e., they are all same.

Notice also that we never assumed θ was smaller than 180°. Thus, (7) is valid when θ is bigger than 180°. See Fig. 5. Then, if we call the outer angle of $\angle ADC$ by θ', we must have $\angle AFC = \theta'/2$. Given this, note that $\theta' = 360° - \theta$. Then, from (7), we obtain (**Problem 1.** Check this!)

$$\angle AFC = 180° - \theta/2 = 180° - \angle ABC \tag{9}$$

Now, we could as well not draw $\angle ADC$ as in Fig. 6. Nevertheless, (9) still remains true. In other words,

$$\angle AFC + \angle ABC = 180° \tag{10}$$

is always satisfied, if the four vertices of a quadrilateral lie on a cicle. We call such a

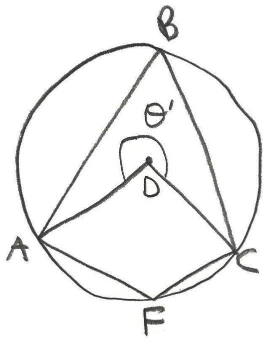

Figure 5: $\angle ADC = \theta'$

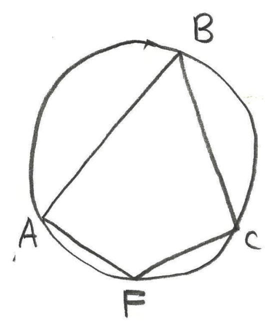

Figure 6: $\angle AFC + \angle ABC = 180°$

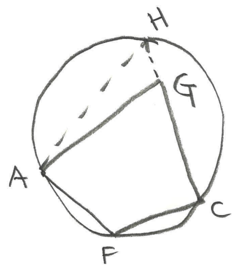

Figure 7: $\angle AFC + \angle AGC > 180°$

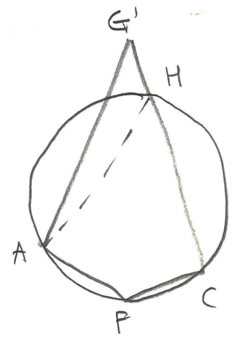

Figure 8: $\angle AFC + \angle AG'C < 180°$

quadrilateral, "inscribed quadrilateral in a circle." Here, we see that the two facing angles of such a quadrilateral always add up to 180°. Thus, it goes without saying that $\angle BAF + \angle BCF = 180°$. Of course, we could have alternatively obtained this from (10), additionally using the fact that the four angles of a quadrilateral always add up to 360°.

From now on, we will show that the converse is true. In other words, if the two facing angles of a quadrilateral add up to 180°, it can be inscribed in a circlr. As we showed in the last article, given a triangle, you can always draw a unique circle in which the triangle is inscribed. In other words, given three points, we can lways draw a unique circle that passes all these three points. See Fig. 7. There is $\square AFCG$ and we have drawn a circle that passes A, F, C. Then, we can extend $\overline{CG}$ to $\overline{CH}$, so that H is on the circle. Then, we see that

$$\angle AFC + \angle AHC = 180° \tag{11}$$

However, we see that $\angle AGC > \angle AHC$. Thus, we see that

$$\angle AFC + \angle AGC > 180° \tag{12}$$

Thus, we see that $\square AFCG$ can't be inscribed in a circle, if (12) is satisfied. Similarly, from Fig. 8, we see that if

$$\angle AFC + \angle AG'C < 180° \tag{13}$$

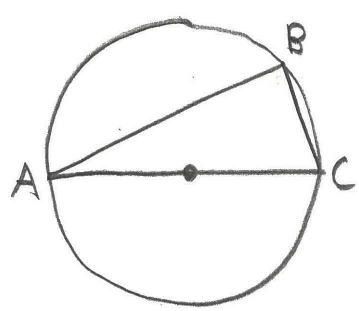

Figure 9: Problem 2. $\angle ABC =?$

is satisfied $\square AFCG'$ can't be inscribed in a circle. Therefore, if and only if

$$\angle AFC + \angle AGC = 180^\circ \tag{14}$$

is satisfied, $\square AFCG$ is inscribed in a circle.

Problem 2. See Fig. 9. $\overline{AC}$ is a diameter of the circle. Then, what is $\angle ABC$?

Summary

- If a quadrilateral is inscribed in a circle, its two angles facing each other always add up to 180°.
- If the sum of two angles facing each other in a quadrilateral is 180°, the quadrilateral can be inscribed in a circle.