Linear equations

Let's say Jorian ate a certain number of apples yesterday and one apple today. And let's say that this amounts to a total of four apples. How many apples did he eat yesterday? This is easy. $4 - 1 = 3$. 3 apples. Is there a fancier way of solving this problem?

Yes, there is. Let's say x is the number of apples Jorian ate yesterday. Then, he ate a total of $x + 1$ apples. On the other hand, we know that this is equal to 4 apples. So, we can write:

$$x + 1 = 4 \tag{1}$$

Now comes the important trick. Since the left-hand side is equal to the right-hand side, if we add a same number to both sides (or subtract the same number from both sides), the equality holds. For example, if $10 = 10$, and we add 5 to both sides, we have $10 + 5 = 10 + 5$ and the equality holds. Similarly, if we subtract 3, we have $10 - 3 = 10 - 3$, and again we see that the equality holds.

Let's use this trick. Let's subtract 1 from the both sides of (1). Then, we have:

$$\begin{aligned} (x + 1) - 1 &= 4 - 1 \\ x + (1 - 1) &= 3 \\ x &= 3 \qquad (2) \end{aligned}$$

So we got the answer. We see here that we subtracted 1 to cancel the 1 on the left-hand side.

Let me give you another equation to solve. This time, the trick is that the equality of an equation still holds if you multiply the both sides by the same number (or divide both sides by the same number.) (For example, if $12 = 12$, $12 \times 3 = 12 \times 3$ and $12 \div 4 = 12 \div 4$.) Here is the equation.

$$\begin{aligned} 4x &= 12 \\ 4x/4 &= 12/4 \\ x &= 3 \qquad (3) \end{aligned}$$

You see here that I have divided by 4, to cancel the 4 in front of x.

Sometimes, you need to use both tricks. For example,

$$\begin{aligned} 7x + 12 &= 19 \\ 7x + 12 - 12 &= 19 - 12 \\ 7x &= 7 \end{aligned}$$

$$\begin{aligned} 7x/7 &= 7/7 \\ x &= 1 \end{aligned} \tag{4}$$

Now, let us present a more difficult type of equation and its solution. You will see here the real power of the tricks I introduced.

$$\begin{aligned} 2x+3 &= 4x-1 \\ 2x+3+1 &= 4x-1+1 \\ 2x+4 &= 4x \\ 2x+4-2x &= 4x-2x \\ 4 &= 2x \\ 4/2 &= 2x/2 \\ 2 &= x \end{aligned} \tag{5}$$

Notice that in this example x is present in both sides, different from earlier examples. This is the reason the equation is more difficult. In the solution, you see that we add 1 to get rid of -1 in the right-hand side. You also see that we, then, subtract $2x$ to get rid of the presence of x in the left-hand side. Here, I emphasize, that in solving this kind of equation, in which x appears in both sides, you have to get rid of the presence of x in one of the sides. Also, as I mentioned similarly in the last article, once you get used to this kind of calculation, you can skip some steps and solve perhaps like the following:

$$\begin{aligned} 2x+3 &= 4x-1 \\ 2x+3+1 &= 4x \\ 2x+4 &= 4x \\ 4 &= 4x-2x \\ 4 &= 2x \\ 4/2 &= x \\ 2 &= x \end{aligned} \tag{6}$$

We see here that subtracting 1 became adding 1 when it went from the right-hand side to the left-hand side. Similarly, adding $2x$ became subtracting $2x$ when it went from the left-hand side to the right-hand side. In other words, subtraction becomes addition and addition becomes subtraction when switching the sides. This is a convenient rule to use. Similarly, multiplication becomes division and division becomes multiplication when switching the side. In our example, multiplying by 2 on the right-hand side became dividing by 2 on the left-hand side.

Sometimes, it's possible that even when other unknowns than x are present in the equation, and solve it for x in terms of the other unknowns. For example, let's solve the following equation. Here n and m are the other unknowns.

$$\begin{aligned} nx + 2 &= mx + 3 \\ nx - mx &= 3 - 2 \\ (n-m)x &= 1 \\ x &= \frac{1}{n-m} \qquad (7) \end{aligned}$$

You see that we treated the other unknowns as ordinary numbers with which we can add, subtract, multiply and divide.

Now, some actual problems that can be solved by using linear equations. In the 12th century, an Indian mathematician and astronomer Bhaskara II wrote a mathematics book "Lilavati" for his daughter Lilavati. The following problem is from Lilavati.

> "Out of a bunch of pure lotus flowers one-third, one-fifth and one-sixth were offered respectively to the gods Shiva, Vishnu, and the Sun; one-fourth was presented to Bhavani. The remaining six flowers were given to the venerable preceptor. Now, Lilavati, tell me quickly how many lotus flowers were there in the bunch."

Let's solve this problem. If we denote the number of lotus flowers in the bunch by x, we have

$$\frac{x}{3} + \frac{x}{5} + \frac{x}{6} + \frac{x}{4} + 6 = x \qquad (8)$$

Thus, we have

$$\begin{aligned} \frac{19}{20}x + 6 &= x \qquad (9) \\ 6 &= x - \frac{19}{20}x \\ 6 &= \frac{x}{20} \\ 120 &= x \qquad (10) \end{aligned}$$

So, there were 120 lotus flowers in the bunch.

Another problem. Let's say that Achilles is in a footrace with a tortoise. When the race started, the tortoise had a head start of 100 meters. If Achilles runs 10 m/s and the tortoise 1 m/s, how long will it take for Achilles to catch up the tortoise, and how far away will they be from the starting point?

Solution. After t seconds, Achilles will be $10t$ meter away from the starting point, and the tortoise will be $100 + t$ meter away from the starting point. When Achilles catches up the tortoise, they are at the same position. Thus,

$$10t = 100 + t \qquad (11)$$

You get $t = 100/9 = 11.111\cdots$. If you plug this into the above equation, Achilles will catch up the tortoise, when they are $1000/9 = 111.11\cdots$ meters from the starting point. We will solve this problem again in our article "Geometric series."

Problem 1. Solve the following problem also from Lilavati.[1]

> "O friend! One-sixth of the bees in a colony entered a patali flower, one-third went to kadamba tree, one-fourth flew to a mango tree and one-fifth went to a tree blooming with campaka flowers. One-thirtieth went to a beautiful bed of lotuses bloomed by the Sun's rays. If only one bee was roving about, how many bees were there in the colony?"

Problem 2. Yesterday, Lina chewed some pieces of gum, and today, she chew two more pieces of gum than yesterday. If she chewed total of ten pieces of gum yesterday and today, how many pieces of gum did she chew yesterday?

Problem 3. Solve the following equations:

$$\frac{x}{3} + \frac{x}{2} = x - 5, \qquad x = ?$$

$$2(x+3) = -2x, \qquad x = ?$$

Problem 4. Explain why the following equations don't have any solutions. (Hint[2])

$$3x + 3 = 3x + 4, \qquad x = ?$$

$$2x + 4 - x = 2\left(\frac{x}{2} + 3\right), \qquad x = ?$$

Problem 5. Explain why any number can be a solution to the following equation. (Hint[3])

$$2(x+2) = 2x + 4$$

An expression like this, which is always satisfied regardless of the values of variables, is called "identity." Here are some other examples of identities:

$$2(x-y) = 2x - 2y, \qquad 2x + 3x - 4x = x \tag{12}$$

Problem 6. Obtain x in terms of n.

$$(n+3)x + 1 = (-2n-9)x + 5$$

What must be n in order that the above equation has no solution? (Hint: Remember how you solved Problem 4. Use the same strategy.)

Problem 7. Obtain x.

$$(n+2)x + 3 = (2n-4)x + 3$$

[1] English translation from "Lilavati of Bhaskaracarya: A Treatise of Mathematics of Vedic Tradition" by Krishnaji Shankara Patwardhan, Somashekhara Amrita Naimpally and Shyam Lal Singh.

[2] If you take the steps that I explained earlier, you will get something like $0x = 1$. Since $0x$, which is always 0, is not equal to 1, we conclude the equality in the original equation cannot be satisfied.

[3] If you take the steps that I explained earlier, you will get something like $0x = 0$. Since this is satisfied for any x, we conclude the equality in the original equation holds for any x. Another way of seeing this is that if you expand the left-hand side, you simply get the right-hand side.

For what value of n does the above equation has any number as its solutions for x? Is it possible that the above equation has no solution for some values of n? (Hint[4])

Problem 8. Obtain x in terms of n and m.

$$(n+1)x + 1 + m = (-3n - 6)x + 4$$

What must be the values for n and m if any x is a solution? How about if there is no solution?

Summary

- Let's say the left-hand side of an equation is equal to the right-hand side of an equation. Then, the equality still holds, even if you add or subtract a same number to both sides. Similarly, the equality of an equation still holds if you multiply or divide both sides by a same number.
- We can use these properties to solve a linear equation.
- When solving an equation, in which x appears in both sides, you have to get rid of the presence of x in one of the sides.
- An expression that is always satisfied is called "identity."
- If an equation can be reduced to the equation of the form $0x = c$, where c is not equal to 0, there is no solution to the original equation.
- If an equation can be reduced to the equation of the form $0x = 0$ without multiplying zero on both sides, then any x is the solution to the original equation.

[4]If you cannot change the above equation to the form $0x = a$ for some non-zero a, this is not possible.