Linear inequalities

In the last article, we have explained how to solve linear equations. We have seen that we can solve linear equations by using the property that the equality still holds even when we add, subtract, multiply, and divide the both sides by the same number.

In this article, we will explain how to solve linear inequalities. The strategy is similar, but we have to be more careful; while the inequality doesn't change when you add or subtract by the same number, or multiply or divide by the same number that is positive, the inequality *changes* the direction when you multiply or divide by a negative number. Let me explain all these with examples.

First, addition by the same number. If we have $5 > 3$, we have $5 + 2 > 3 + 2$. We can add a negative number, too. $5 + (-2) > 3 + (-2)$. Similarly, if we have $3 < 5$, we have $3 + 2 < 5 + 2$ and $3 + (-2) < 5 + (-2)$.

Second, subtraction by the same number. If we have $5 > 3$, we have $5 - 4 > 3 - 4$. We can also subtract a negative number. $5 - (-1) > 3 - (-1)$. Similarly, if we have $3 < 5$, we have $3 - 4 < 5 - 4$ and $3 - (-1) < 5 - (-1)$.

Third, multiplication by a positive number. If we have $5 > 3$, we have $5 \times 2 > 3 \times 2$. Similarly, if we have $3 < 5$, we have $3 < 5 \times 2$.

Fourth, division by a positive number. If we have $5 > 3$, we have $5 \div 2 > 3 \div 2$. Similarly, if we have $3 < 5$, we have $3 \div 2 < 5 \div 2$.

Now comes the important point. Let's see what happens if we multiply a negative number, -2 on $5 > 3$. We see that $5 \times (-2) > 3 \times (-2)$ is *not* satisfied. Clearly, -10 is smaller than -6. Thus, we have $5 \times (-2) < 3 \times (-2)$. So, we see that multiplication by a negative number changes the inequality. Similarly, if we have $3 < 5$, $3 \times (-2) > 5 \times (-2)$. -6 is bigger than -10.

The inequality also changes when you divide by a negative number, because dividing by a negative number is equivalent multiplying by (another) negative number. For example, dividing by -2 is equivalent to multiplying by $-1/2$.

Let me explain more concretely why the inequality sign changes when we multiply a negative number. We start with

$$5 > 3 \tag{1}$$

Remember that the inequality doesn't change if we subtract the same number. So, let's subtract the number of the left-hand side, then the number on the right-hand side

$$\begin{aligned} 5 - 5 &> 3 - 5 \quad (2) \\ 0 &> 3 - 5 \quad (3) \end{aligned}$$

$$\begin{aligned} 0-3 &> 3-5-3 = (3-3)-5 && (4) \\ 0-3 &> 0-5 && (5) \\ -3 &> -5 && (6) \end{aligned}$$

As $a > b$ implies $b < a$, we finally have

$$-5 < -3 \tag{7}$$

In other words, we started out with $x > y$ and ended up with $-x < -y$. More generally,

$$\begin{aligned} x &> y && (8) \\ x-x &> y-x && (9) \\ 0 &> y-x && (10) \\ 0-y &> y-x-y && (11) \\ -y &> -x && (12) \\ -x &< -y && (13) \end{aligned}$$

Could we derive this faster? Remember the trick that addition becomes subtraction and subtraction becomes addition, if you move terms across the equality sign. The same can be said about the inequality sign, because this trick is based on our facts mentioned earlier that the inequality doesn't change if you add or subtract the same number.

$$x > y \tag{14}$$

If you move x to the right side,

$$0 > y - x \tag{15}$$

If you move y to the left side,

$$-y > -x \tag{16}$$

Finally, as $a > b$ implies $b < a$,

$$-x < -y \tag{17}$$

To repeat, we started with $x > y$, and ended up with $-x < -y$. We indeed see that multiplying by -1 changes the inequality. Notice that this is satisfied regardless of the sign of x and y, as we never assumed the specific signs of x and y in our derivation.

Nevertheless, it is instructive to check it case by case. For example, if x and y are positive, we have $5 > 3$ implies $-5 < -3$. If x is positive and y is negative, we have $2 > -4$ implies $-2 < 4$. If x and y are both negative, we have $-4 < -3$ implies $4 > 3$.

So far, we checked that the inequality changes when we multiply by -1. How about when we multiply other negative numbers, such as -2 or -3? We know that multiplying by -2 or -3 is the same thing as multiplying by -1 first, then multiply by 2 or 3. The inequality changes, when we multiply by -1 first. Then, when we multiply 2 or 3, the inequality doesn't change. So, if you multiply -2 or -3, the inequality changes only once. If it sounds somewhat abstract, let's see with an example:

$$\begin{aligned} x &> y & (18) \\ -x &< -y & (19) \\ -x \times 2 &< -y \times 2 & (20) \\ -2x &< -2y & (21) \end{aligned}$$

From the same logic, multiplying a negative number changes the inequality.

Now, some actual examples. Let's solve the following.

$$\begin{aligned} x+3 &< 2x-4 & (22) \\ x-2x &< -4-3 & (23) \\ -x &< -7 & (24) \\ x &> 7 & (25) \end{aligned}$$

Of course, we can solve this slightly differently.

$$\begin{aligned} x+3 &< 2x-4 & (26) \\ 3+4 &< 2x-x & (27) \\ 7 &< x & (28) \\ x &> 7 & (29) \end{aligned}$$

Either way, you get the same answer.

Another example. Let's solve $x+4 < -x+2 < 5$. We need to solve $x+4 < -x+2$ and $-x+2 < 5$. We get $x < -1$ and $x > -3$, respectively. So, the answer is $-3 < x < -1$.

Another example. Let's solve $x < 2x < -x-4$. We need to solve $x < 2x$ and $2x < -x-3$. We get $x > 0$ and $x < -1$. Since there is no x that satisfies both $x > 0$ and $x < -1$, there is no solution to these inequalities.

Final comment. So far, we only talked about $>$ and $<$, but the observations made in this article regarding $>$ and $<$ can be equally applied to $\geq$ and $\leq$. For example, if we have $x \leq y$, we necessarily have $-x \geq -y$.

Problem 1. Solve the following.

$$-x-3 > x-5$$

$$y-4 \geq 3y-12$$

$$5z-4 < 4z+3$$

Problem 2. Solve the following.

$$x+3 < 2x \leq 3x$$

$$y + 4 < 3y < 2y$$

$$4 \leq z < -z + 6$$

Problem 3. Solve the following.

$$|x| < 4$$

$$|-a| < 4$$

Problem 4. Solve the following. (Hint[1])

$$|y - 3| < 4$$

$$|5 - z| < 4$$

Problem 5. If $a < b$ and $1/a > 1/b$, what can we say about the signs of a and b? (Hint[2])

Problem 6. This problem is about Seiberg duality. Of course, you need to know very advanced physics (supersymmetry) to understand Seiberg duality, but you can still check a couple of the evidences for Seiberg duality with very easy math. Seiberg duality states that a theory with the number of flavor N_f with the number of color N_c and its dual theory (which we denote with $'$) with the number of flavor $N_f' = N_f$ and the number of color $N_c' = N_f - N_c$ are the same theory in low energy. (If you are curious about what flavor and color mean, you can read "Pauli's exclusion principle, Color Charge of Quarks, Asymptotic freedom and Confinement" but it won't help you to solve this problem at all.) First, check that $N_f = N_f'$ and $N_c = N_f' - N_c'$ are satisfied. This means that the dual theory of the dual theory is the original theory. The original theory has an infrared fixed point under the following condition:

$$\frac{3}{2}N_c < N_f < 3N_c \tag{30}$$

Similarly, the dual theory of the original theory has infrared fixed point under the following condition:

$$\frac{3}{2}N_c' < N_f' < 3N_c' \tag{31}$$

Check that this condition is the same condition as (30). In other words, the dual theory has infrared fixed point if and only if the original theory has infrared fixed point. Of course, the demonstration of Seiberg duality is not so easy as presented in this problem, as these are not the only evidences for Seiberg duality. I wonder how Nathan Seiberg first discovered Seiberg duality, as he came up with $N_f' = N_f$ and $N_c' = N_f - N_c$ out of nowhere, then demonstrated that everything fitted perfectly.

Summary

- A linear inequality can be solved by using a strategy that you use when you solve linear equations.

[1] Problem 3 can be helpful.

[2] Divide $a < b$ by ab.

- However, you have to be careful. While the inequality doesn't change when you add or subtract by the same number, or multiply or divide by the same number that is positive, the inequality *changes* the direction when you multiply or divide by a negative number.
- To solve inequalities of form $a < b < c$, you need to solve $a < b$ and $b < c$, and find the overlapping range of the two solutions.