Multiplication and division of negative numbers

Having learned how to add and subtract negative numbers, let me now explain how to multiply and divide negative numbers. First, let's recall how to multiply positive numbers. We have

$$3 + 3 + 3 + 3 = 12, \quad \text{which implies } 3 \times 4 = 12 \tag{1}$$

By the same token,

$$(-3) + (-3) + (-3) + (-3) = -12, \quad \text{implies } (-3) \times 4 = -12 \tag{2}$$

In other words, $(-a) \times b = -(a \times b)$. Notice that if you multiply a negative number by a positive number, you get a negative number. Now, remember that the multiplication is always commutative. For example, $3 \times 4 = 4 \times 3$. Therefore, we expect $(-3) \times 4 = 4 \times (-3)$, from which we conclude

$$4 \times (-3) = -12 \tag{3}$$

Notice that if you multiply a positive number by a negative number, you get a negative number.

Now, some applications. Suppose you have total of 50 dollars today. If you have been earning and will earn 2 dollars everyday, how much total money will you have 4 days from now, assuming that you haven't been spending and won't spend any money? It's easy. $50 + 2 \times 4 = 58$ dollars. Another question: How much money did you have 3 days ago? You earned $2 \times 3 = 6$ dollars last 3 days. So, you had only $50 - 6 = 44$ dollars 3 days ago. Another way of seeing is that 3 days ago is equal to "-3" days from now. So, -3 days from now, you had $50 + 2 \times (-3) = 44$ dollars. In either way, you get the correct answer.

Suppose you have 40 dollars today. If you have been spending and will spend 3 dollars everyday, how much money will you have 4 days from now? You will spend $3 \times 4 = 12$ dollars during next 4 days. Thus, you will have $40 - 12 = 28$ dollars remaining. Another way of seeing this is you earn -3 dollars every day. Thus, we get $40 + (-3) \times 4 = 40 + (-12) = 28$ dollars. Another question. How much money did you have 5 days ago? You spent $3 \times 5 = 15$ dollars last 5 days. So, you had 15 dollars more 5 days ago. So, you had $40 + 15 = 55$ dollars 5 days ago. Another way of seeing this is you earn -3 dollars every day, and -5 days from now, you had $40 + (-3) \times (-5) = 55$ dollars. So, we conclude

$$(-3) \times (-5) = 15 \tag{4}$$

In other words, $(-a) \times (-b) = a \times b$. Notice that if you multiply a negative number by a negative number, you get a positive number.

It may be somewhat unexpected, so let me give you another way of proving this. We know that $(a + b) \times c = a \times c + b \times c$. Consider, now

$$(3 + (-3)) \times (-5) = 0 \times (-5) = 0 \tag{5}$$

On the other hand, the above is equal to

$$0 = 3 \times (-5) + (-3) \times (-5) = -15 + (-3) \times (-5) \tag{6}$$

Since you have to add 15 to -15 to get 0, we conclude that (4) must be correct.

Having considered the multiplication of negative numbers, now let's consider the division of negative numbers. We know that $a \times b = c$ implies $c \div b = a$. Thus, we have

$$(-3) \times 4 = -12 \qquad \text{implies} \quad (-12) \div 4 = -3 \tag{7}$$

$$4 \times (-3) = -12 \qquad \text{implies} \quad (-12) \div (-3) = 4 \tag{8}$$

$$(-3) \times (-5) = 15 \qquad \text{implies} \quad 15 \div (-5) = -3 \tag{9}$$

Summarizing, if you divide a negative number by a positive number, you get a negative number. If you divide a negative number by a negative number, you get a positive number. If you divide a positive number by a negative number, you get a negative number.

If you denote a positive number by $(+)$, and a negative number by $(-)$. What we have learned in this article can be summarized as follows.

$$(+) \times (+) = (+) \tag{10}$$

$$(+) \times (-) = (-) \tag{11}$$

$$(-) \times (+) = (-) \tag{12}$$

$$(-) \times (-) = (+) \tag{13}$$

$$(+) \div (+) = (+) \tag{14}$$

$$(+) \div (-) = (-) \tag{15}$$

$$(-) \div (+) = (-) \tag{16}$$

$$(-) \div (-) = (+) \tag{17}$$

Notice that for both multiplication and division, if you have no negative sign on the left-hand side, the result is positive, if you have one negative sign on the left-hand side, the result is negative, and if you have two negative sign on the left-hand side, the result is positive.

Can we extend these lists of the sign rules for two numbers to three numbers? Yes. All that matters is the number of negative signs, since multiplying a positive number doesn't change the sign. Also, if there are three negative signs, it is negative, because two negative signs are positive, and if you have an extra negative sign, it's negative. Thus, we have

- 0 negative sign: positive

- 1 negative sign: negative
- 2 negative signs: positive
- 3 negative signs: negative

Can we extend these lists of the sign rules to arbitrary numbers of multiplication and division? Yes. Whenever you have one more negative sign, the sign flips. For example, if there is one negative sign, it's negative. If you have one more negative sign, there are total of two negative signs, and it's now positive. The sign flipped from negative to positive. Thus, we can extend the above list. For four negative signs, it is positive, as three negative signs is negative, and if you have one more negative sign, the sign flips and it's positive. Thus, if we extend the above list this way, we get

- 4 negative signs: positive
- 5 negative signs: negative
- 6 negative signs: positive
- 7 negative signs: negative

and so on. Do you see the pattern? If there are even number of negative signs, it is positive, and if there are odd number of negative signs, it is negative.

Another way of seeing this is that if you have an even number of negative signs, you can pair the negative signs, then each pair will be positive. So, the end result will be positive.

If you have an odd number of negative signs, and if you try to pair them up, one will be left out. So, the end result will be negative.

Now, you can understand the old lore of physics.

> The difference between a good physicist and a bad physicist is that a good physicist makes an even number of sign mistakes.

For example, if you are to calculate the following

$$(-3) \times (-1) \times (-4) \div 2 \div (-2) \times (-3) \times 1 \times (-1) \times (-2) \tag{18}$$

there are seven negative signs, so the answer is positive, and as $3\times1\times4\div2\div2\times3\times1\times1\times2 = 18$, the final answer is -18. However, if you make a mistake by ignoring one negative sign, then you will get the wrong answer "18." Then, you are a bad physicist. On the other hand, if you make a mistake by ignoring even number of negative signs, say 2, then you will think that there are five number of negative signs, and you will still get -18. Then, you are a good physicist.

Finally, I would like to introduce the concept of "absolute value." The absolute value of a number is defined by the distance between that number and 0. See Fig. 1. The absolute value of 2 is 2. The distance between 2 and 0 is $2 - 0 = 2$. Similarly, the absolute value of 1 is 1, the absolute value of 3.7 is 3.7, and so on. In other words, the absolute value of a positive number is the positive number itself. If we denote the absolute value by $||$, for a positive number x, we have

$$|x| = x, \qquad \text{for } x > 0 \tag{19}$$

What about the absolute value of a negative number? See Fig 2. The absolute value of -3 is 3, as the distance between 0 and -3 is 3 as $0-(-3)=3$. Similarly, the absolute value of -4 is 4, the absolute value of -1 is 1, and so on. In other words, the absolute value of a negative number x is given by $0 - x = -x$. In other words,

$$|x| = -x, \qquad \text{for } x < 0 \tag{20}$$

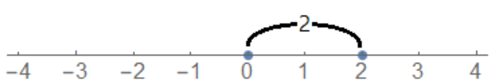

Figure 1: $|2| = 2$

Figure 2: $|-3| = 3$

What is the absolute value of 0? The distance between 0 and 0 is 0. Thus,

$$0 = 0 \tag{21}$$

Summarizing, we can think of the absolute value as "taking away" the negative sign. See below for more examples. You see that the absolute value does nothing for non-negative numbers, while it takes away the negative sign for negative numbers.

$$|4| = 4, \qquad |3| = 3, \qquad |-4.5| = 4.5, \qquad |0| = 0, \qquad |-2| = 2 \tag{22}$$

Notice also that the absolute value of any number can never be negative.

Let me conclude this article with a comment. Mathematicians began to have ideas on negative numbers around the 15th century, and the concept of negative numbers was firmly established in the 17th century. At this point, you may not be able to appreciate the importance of negative number. It may just seem as a convenient way of expressing "debt" as we have seen in the last article. However, in our essay "The power of negative number" you will see that the concept of negative numbers is more than a convenient tool.

Problem 1.

$$(-2) \times 4 = ?, \qquad 4 \times (-3) = ?, \qquad 5 \times (-2) = ?$$

Problem 2.

$$(-2) \times (-4) = ?, \qquad (-4) \times (-5) = ?, \qquad (-1) \times (-3) = ?$$

Problem 3.

$$(-3) \times (-4) = ?, \qquad (-4) \times (-7) = ?, \qquad (-1) \times 3 = ?$$

Problem 4.

$$(-20) \div 5 = ?, \qquad (-20) \div 4 = ?, \qquad (-14) \div 2 = ?$$

Problem 5.

$$(-3) \div (-1) = ?, \qquad (-3) \div (-3) = ?, \qquad (-4) \div (-2) = ?$$

Problem 6.

$$1 \div (-1) = ?, \qquad 3 \div (-3) = ?, \qquad 4 \div (-2) = ?$$

Problem 7.

$$8 \div (-2) = ?, \qquad (-9) \div 3 = ?, \qquad (-8) \div (-2) = ?$$

Problem 8.

$$2 \times (-2) \times (-2) = ?, \qquad (-1) \times (-3) \times (-2) = ?$$

Problem 9.

$$|5 - 8| = ?, \qquad |5| - |-8| = ?, \qquad |-3 + 2| = ?, \qquad |-3| + |2| = ?$$

Problem 10.

$$|(-1) \times (-2) \times (-3)| = ?, \qquad |2 - 5| = ?$$