Polynomials, expansion and factoring

A polynomial is an expression that can be written as $ax_1^{n1}x_2^{n_2}x_3^{n_3}\cdots x_m^{n_m}$ or its sum, where a is a number, often called "coefficient," xs are variables, and ns are non-negative integers. For example, the following expressions are all polynomials:

$$xy^3, \qquad \frac{1}{2}, \qquad 2y^2z - 3xyz - x + 4, \qquad 3x^2yz + 4y^2z, \qquad 5x + 3$$

xy^3 is a polynomial, because we can set $a = 1$, $x_1 = x$, $n_1 = 1$, $x_2 = y$, $n_2 = 3$. $\frac{1}{2}$ is a polynomial, because we can set $a = 1/2$, $n_1 = 0$. $2y^2z - 3xyz - x$ is a polynomial as each term in the expression, i.e., $2y^2z$, $-3xyz$, $-x$, 4 is a polynomial. In this expression, we see that the coefficient for y^2z is 2, the coefficient for xyz is -3, the coefficient for x is -1. The last term in this expression, 4, which does not involve any variable is called the "constant term." As another example, the constant term in $5x + 3$ is 3.

On the other hand, the following expressions are *not* polynomials:

$$\frac{3x}{y}, \qquad xy - 2x^2 + \sqrt{x}, \qquad \frac{3xz}{2x^2 + 5}, \qquad x^2 + 3\sqrt{x}$$

The first expression, which is equal to $3xy^{-1}$ is not a polynomial, because y has a negative exponent. The second expression is not a polynomial, because the last term, which is equal to $x^{1/2}$ has a non-integer exponent. The third expression is not a polynomial, because it cannot be expressed as the sum of desired form that we mentioned.

Problem 1. Which one of these are polynomials and which are not?

$$\frac{3x^2 + 4x + 5y}{10}, \qquad 10x + 4y, \qquad \sqrt{xy}, \qquad 4 + 5x + \sqrt{x}$$

The degree of a term is the sum of the exponents in a term. For example, the degree of xy^3 is 4 as $xy^3 = x^1y^3$ and $1 + 3 = 4$. Similarly, the degree of $-3x^4yz$ is $4 + 1 + 1 = 6$.

The degree of a polynomial is the largest of the degrees of each term. For example, $2xy^2z - 4wxyz^3$ has a degree 6 as $2xy^2z$ has a degree 4 and $-4wxyz^3$ has a degree 6. Expressions which have degree 1 are called "linear." For example $x + y = 4$ and $x - y = -3$ are considered as linear equations.

Problem 2. Find the degree of the following polynomials

$$5xyzw + 4x, \qquad x - y - zw, \qquad z^5 + 4x^2 - 3x, \qquad \frac{3}{2}, \qquad x + 2 - 3x^2$$

Polynomials can be added, subtracted, and multiplied. The result of this are still polynomials. First, examples for addition and subtraction.

$$(xy + 4xy^2) + (-xy + 3xy^2) = 7xy^2 \qquad (1)$$

$$(6y + 7xyz) - (5y - 3xyz + 2) = 6y + 7xyz - 5y + 3xyz - 2 = y + 10xyz - 2 \tag{2}$$

In these examples, notice that similar terms are added (or subtracted) together. In (1), we have

$$xy + (-xy) = (1 + (-1))xy = 0xy = 0 \tag{3}$$

$$4xy^2 + 3xy^2 = (4 + 3)xy^2 = 7xy^2 \tag{4}$$

Similarly, in (2),

$$6y - 5y = (6 - 5)y = 1y = y \tag{5}$$

$$7xyz + 3xyz = (7 + 3)xyz = 10xyz \tag{6}$$

As mentioned in our earlier article "Symbols and Expressions," remember also that the negative sign in front of $(5y - 3xyz + 2)$ applies to each term in the parenthesis. Therefore,

$$-(5y - 3xyz + 2) = -5y - (-3xyz) - (+2) = -5y + 3xyz - 2 \tag{7}$$

It is *not* equal to $-5y - 3yz + 2$.

Now, an example for multiplication.

$$(x + y)(z + w) = (x + y)z + (x + y)w = xz + yz + xw + yw \tag{8}$$

where in the last equation we used the distributive property.

However, when a polynomial is divided by another polynomial, the result can be a non-polynomial. For example, the following is not polynomial,

$$(xy + yz) \div y^2 = xy/y^2 + yz/y^2 = \frac{x}{y} + \frac{z}{y} \tag{9}$$

as the above expression has negative exponents (i.e. $xy^{-1} + zy^{-1}$)

Now comes an important concept. An expansion of a product of sums expresses it as a sum of products by using the distributive property. In other words, you remove the parenthesis. (8) is a good example. Here comes some important expansions.

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b) \\ &= (a + b)a + (a + b)b \\ &= a^2 + ba + ab + b^2 \\ &= a^2 + 2ab + b^2 \end{aligned} \tag{10}$$

$$\begin{aligned}(a - b)^2 &= (a - b)(a - b) \\ &= (a - b)a - b(a - b) \\ &= a^2 - ba - ab + b^2 \\ &= a^2 - 2ab + b^2 \end{aligned} \tag{11}$$

$$
\begin{aligned}
(a+b)(a-b) &= (a+b)a - b(a+b) \\
&= a^2 + ab - ba - b^2 \\
&= a^2 - b^2
\end{aligned}
\tag{12}
$$

I recommend you memorize these three formulas. These formulas come handy, when you expand expressions of this form. For example, you can expand

$$(x+3)^2 = x^2 + 2(3x) + 3^2 = x^2 + 6x + 9 \tag{13}$$

As another example,

$$(x+2y)^2 = x^2 + 2(x \cdot 2y) + (2y)^2 = x^2 + 4xy + 4y^2 \tag{14}$$

Actually, you can regard (11) as an example of (10) if you think along the following way:

$$(a+(-b))^2 = a^2 + 2(a)(-b) + (-b)^2 = a^2 - 2ab + b^2 \tag{15}$$

Using the distributive property iteratively, one can expand more complicated expressions. For example,

$$
\begin{aligned}
(a+b)^3 &= (a+b)(a+b)^2 \\
&= (a+b)(a^2 + 2ab + b^2) \\
&= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2) \\
&= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 \\
&= a^3 + 3a^2b + 3ab^2 + b^3
\end{aligned}
\tag{16}
$$

Problem 3. Expand the following.

$$(x+2)^2 = ?, \quad (x-2)^2 = ?, \quad (x-1)^2 = ?, \quad (x+5)^2 = ?$$

$$(x+2y)^2 = ?, \quad (2x+1)^2 = ?, \quad (2x-1)^2 = ?, \quad (2x+3)^2 = ?$$

$$(2a+b)^2 = ?, \quad (y-x)^2 = ?, \quad (x+1)^3 = ?, \quad (x-1)^3 = ?$$

$$(x-4)(x+4) = ?, \quad (2x-y)(2x+y) = ?, \quad (x+10)(x-10) = ?$$

Problem 4. Check the following expansion, which I recommend you memorize.

$$(x+a)(x+b) = x^2 + (a+b)x + ab, \qquad (x-a)(x-b) = x^2 - (a+b)x + ab \tag{17}$$

We can use the above formulas to expand certain expansions conveniently. For example,

$$(x-4)(x+3) = (x+(-4))(x+3) = x^2 + (-4+3)x + (-4)3 = x^2 - x - 12 \tag{18}$$

Notice that the second expression in (17) is a special case of the first expansion. Let's see this:

$$(x+(-a))(x+(-b)) = x^2 + ((-a)+(-b))x + (-a)(-b) = x^2 - (a+b)x + ab \tag{19}$$

Problem 5. Expand the following.

$$(x+3)(x+2) =?, \qquad (x+1)(x+4) =?, \qquad (x+1)(x+2) =?$$

$$(x-1)(x-3)-?, \qquad (x-4)(x-5) =?, \qquad (x-4)(x-1) =?$$

$$(x+2)(x-3) =?, \qquad (x-3)(x+4) =?, \qquad (x-1)(x+2) =?$$

$$(a-b)^3 =?, \qquad (a+b)(a^2-ab+b^2) =?, \qquad (a-b)(a^2+ab+b^2) =?$$

$$(x^2+1)(x+1) =?, \qquad (x^2-1)(2x+1) =?, \qquad (x^2-2x)(x^2+2x) =?$$

$$(3x+4)(2x+1) =?, \qquad (3x+1)(x-2) =?, \qquad (x+1)(4x-1) =?$$

There is a trick to expand the expression of the form in the last line, i.e., $(ax+b)(cy+d)$. Let me explain with one of the examples above. Let's say you expand

$$(3x+4)(2x+1) = \tag{20}$$

First, you multiply terms proportional to x. By multiplying $3x$ and $2x$, you get $6x^2$. Then, to obtain the term proportional to x, you consider the following multiplications and add them.

$$(3x+4)(2x+1) \tag{21}$$

i.e.,

$$3x \times 1 + 4 \times 2x = 11x \tag{22}$$

So, $11x$. Finally, if you multiply 4 and 1, you get 4. Therefore the answer is $6x^2+11x+4$. It is not a great trick, but convenient. If you cannot remember this trick, you can simply expand by the usual way.

Problem 6. Expand the following.

$$(2x+1)(x+1) =?, \qquad (2x-1)(x-1) =?, \qquad (3x+1)(x-1) =?$$

$$(5x-1)(4x-1) =?, \qquad (x-4)(x+5) =?, \qquad (3x+1)(x+1) =?$$

On the other hand, factoring is the reverse process of expansion. For example,

$$2x+4y = 2(x+2y), \qquad x^2-4y^2 = (x+2y)(x-2y), \qquad x^2+8x+16 = (x+4)^2 \tag{23}$$

A good strategy to factor is finding the common factor among the terms. For example, to factor $2ma+mb+mc$, notice that each term has the common factor m. Thus,

$$2ma+mb+mc = m(2a+b+c) \tag{24}$$

In some cases, this strategy alone does not work, as you can see in the second and the third examples of (23). Nevertheless, if each term has a common factor, you must factor out by this common factor first. For example,

$$mx^2+2mx+m = m(x^2+2x+1) \tag{25}$$

then, we can take the second step as follow

$$m(x^2+2x+1) = m(x+1)^2 \tag{26}$$

Problem 7. Factor out the following.

$$2xy+4xyz=?, \qquad x^2-4=?, \qquad x^2+2x+1=?, \qquad 3x-9xy=?$$

$$x^2+4x+4=?, \qquad x^2-9=?, \qquad x^2-2x+1=?, \qquad 4x^2-y^2=?$$

$$2xy+4xyzw=?, \qquad 4x^2-9=?, \qquad x^2-4x+4=?, \qquad 9xy-3x=?$$

Now some applications of expansion.

$$(2+\sqrt{3})^2 = 2^2+2(2\sqrt{3})+(\sqrt{3})^2 = 4+4\sqrt{3}+3 = 7+4\sqrt{3} \tag{27}$$

Problem 8. Expand the following expressions.

$$(2-\sqrt{2})^2=?, \qquad (\sqrt{2}+1)^2=?, \qquad (\sqrt{2}+\sqrt{3})^2=?$$

Suppose you want to know a^2+b^2, given that $a+b=5$ and $ab=2$. To obtain the answer, you may want to get a and b first. However, to do so, you would need to know how to solve quadratic equations, which we haven't taught you yet. Nevertheless, there is a way to find the answer without explicitly knowing a and b. Observe $(a+b)^2 = a^2+2ab+b^2$ implies

$$a^2+b^2 = (a+b)^2-2ab \tag{28}$$

Thus, we get

$$a^2+b^2 = 5^2-2\times 2 = 21 \tag{29}$$

Problem 9. Let's say $a-b=5$ and $ab=2$. What is a^2+b^2?

Problem 10. Let's say $a^2+b^2=20$ and $ab=3$, what is $a+b$? What is $a-b$ if $a>b$?

Problem 11. Let's say

$$a+b=10, \qquad ab=5, \qquad a>b \tag{30}$$

What is a^2+b^2? What is $a-b$? Find a and b. (Hint[1])

Problem 12. Prove the following. This is useful in string theory. (We assume $x \neq 0, 1$. Hint[2])

$$x^u(1-x)^{v-1} + x^{u-1}(1-x)^v = x^{u-1}(1-x)^{v-1} \tag{31}$$

Problem 13. What degree of a polynomial do you get if you add a polynomial of degree 3 to a polynomial of degree 2?

Problem 14. What are the possible degrees of the polynomial you get if you add a polynomial of degree 4 to a polynomial of degree 4? (Challenging!)

Probelm 15. What degree of a polynomial do you get if you multiply a polynomial of degree 2 by a polynomial of degree 1? (Challenging!)

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[1] If you know $a+b$ and $a-b$, you can find a and b by solving systems of linear equations.

[2] Use $x^u = x^{u-1}x$ and $(1-x)^v = (1-x)^{v-1}(1-x)$.

Summary

- A polynomial is an expression that can be written as $ax_1^{n1}x_2^{n_2}x_3^{n_3}\cdots x_m^{n_m}$ or its sum, where a is a number, xs are variables, and ns are non-negative integers.
- The degree of a term is the sum of the exponents in a term.
- The degree of a polynomial is the largest of the degrees of each term.
- When polynomials are added, subtracted, or multiplied, the result of this are still polynomials.
- However, when a polynomial is divided by another polynomial, the result can be a non-polynomial.
- An expansion of a product of sums expresses it as a sum of products by using the distributive property. In other words, you remove the parenthesis.
- $(a+b)^2 = a^2 + 2ab + b^2, \qquad (a-b)^2 = a^2 - 2ab + b^2, \qquad (a+b)(a-b) = a^2 - b^2$
- $(x+a)(x+b) = x^2 + (a+b)x + ab, \qquad (x-a)(x-b) = x^2 - (a+b)x + ab$
- Factoring is the reverse process of expansion.