Quadratic equation

You know how to solve the following type of equations.

$$x^2 = 2, \qquad y^2 = 4, \qquad z^2 = 81 \tag{1}$$

The answers are

$$x = \pm\sqrt{2}, \qquad y = \pm 2, \qquad z = \pm 9 \tag{2}$$

where $\pm$ here denotes the fact that the answer can take both positive and negative values. For example, $\pm 2 = 2, -2$.

However, the following type of equations are much more tricky:

$$x^2 + 6x + 5 = 0 \tag{3}$$

In this article, I will teach you how to solve this type of equations, which is called a "quadratic equation." A quadratic equation is in the following form.

$$ax^2 + bx + c = 0 \tag{4}$$

where $a \neq 0$, b, and c are constants.(If $a = 0$, it is just a linear equation, i.e., $bx + c = 0$).

Before solving (3), let's try to solve equations that are more tricky than (1), but less tricky than (3). Let's solve the following equations:

$$(x-2)^2 = 1 \tag{5}$$

It's easy to find what $(x-2)$ is. It is

$$x - 2 = 1 \text{ or } -1 \tag{6}$$

Thus,

$$x = 2 + 1 \text{ or } 2 + (-1) \tag{7}$$

$$x = 3, 1 \tag{8}$$

Another example

$$(x+4)^2 = 3 \tag{9}$$

It is easy to find what $(x + 4)$ is. It is

$$x + 4 = \pm\sqrt{3} \tag{10}$$

Thus,

$$x = -4 \pm \sqrt{3} \tag{11}$$

Another example

$$(x + 3)^2 = 4 \tag{12}$$

It is easy to find what $(x + 3)$ is. It is

$$x + 3 = 2, -2 \tag{13}$$

Thus,

$$\begin{aligned} x &= 2 - 3, \text{ or} - 2 - 3 && (14) \\ x &= -1, -5 && (15) \end{aligned}$$

Now, let's try to solve (3). However, we just solved it! If you expand (12), you get

$$\begin{aligned} x^2 + 6x + 9 = 4 && (16) \\ x^2 + 6x + 9 - 4 = 0 && (17) \\ x^2 + 6x + 5 = 0 && (18) \end{aligned}$$

In other words, (3) and (12) are the same equations. Thus, we get an idea of how to solve quadratic equations. If we have a quadratic equation of the form (3), we need to turn it into the form (12). Then, we can solve it. Let's do it once again. If we begin with $x^2 + 6x + 5 = 0$, we have

$$\begin{aligned} (x^2 + 6x + 9 - 9) + 5 &= 0 \\ (x + 3)^2 - 9 + 5 &= 0 \\ (x + 3)^2 &= 4 \\ x + 3 &= 2, -2 \\ x &= -1, -5 && (19) \end{aligned}$$

Note the crucial step. You added 9 and subtracted 9 to $x^2 + 6x$ to make an expression of the form $(x + 3)^2$. Remember, $(x + e)^2 = x^2 + 2ex + e^2$. In our case we set $e = 3$ because $2ex$ must match $6x$ in (3). So, we set

$$x^2 + 2ex = x^2 + 2ex + e^2 - e^2 = (x + e)^2 - e^2 \tag{20}$$

In other words, we added and subtracted e^2. This whole procedure is known as "completing the square." To repeat, in our example, the coefficient of the term linear in x

was 6. Half of 6 is 3. Thus, you can complete the square as $(x+3)^2$, to absorb the $6x$ term into an expression of a square.

One more example:

$$\begin{aligned}
x^2 - 4x - 3 &= 0 \\
x^2 - 4x + 4 - 4 - 3 &= 0 \\
(x-2)^2 - 7 &= 0 \\
(x-2)^2 &= 7 \\
x - 2 &= \pm\sqrt{7} \\
x &= 2 \pm \sqrt{7} \qquad (21)
\end{aligned}$$

In this example, the coefficient of the term linear in x was -4. As half of -4 is -2. Thus, we can complete the square by $(x-2)^2$ to absorb the term $-4x$.

What if a in (4) is not 1? If we simply divide both sides by a, the coefficient in front of x^2 will be 1. Then, we can just take similar steps as before. For example,

$$\begin{aligned}
6x^2 - 7x - 3 &= 0 \qquad (22) \\
x^2 - \frac{7}{6}x - \frac{1}{2} &= 0 \\
\left(x - \frac{7}{12}\right)^2 - \frac{49}{144} - \frac{1}{2} &= 0 \\
\left(x - \frac{7}{12}\right)^2 &= \frac{121}{144} \\
x - \frac{7}{12} &= \frac{11}{12}, -\frac{11}{12} \\
x &= \frac{3}{2}, -\frac{1}{3} \qquad (23)
\end{aligned}$$

Let me just present one more example:

$$\begin{aligned}
4x^2 + 5x - 3 &= 0 \qquad (24) \\
x^2 + \frac{5}{4}x - \frac{3}{4} &= 0 \\
\left(x + \frac{5}{8}\right)^2 - \frac{25}{64} - \frac{3}{4} &= 0 \\
\left(x + \frac{5}{8}\right)^2 &= \frac{73}{64} \\
x + \frac{5}{8} &= \pm\frac{\sqrt{73}}{8} \\
x &= \frac{-5 \pm \sqrt{73}}{8} \qquad (25)
\end{aligned}$$

So far, we have seen the cases in which there are two solutions. However, it is possible for there to be only one solution or no solutions at all. For example, as you see, the following equation has only one solution.

$$\begin{aligned} x^2 - 4x + 4 &= 0 \quad (26)\\ (x-2)^2 &= 0 \\ x - 2 &= 0 \\ x &= 2 \quad (27) \end{aligned}$$

Notice that in the earlier cases there are two solutions because the equation of type $(x+e)^2 = f$ implies $x + e = \pm\sqrt{f}$. This means that we had two cases, the plus sign and the negative sign, as long as f is positive. However, in the present case, f is 0, which implies that $\pm\sqrt{f}$ is not two numbers, but only one number. $+\sqrt{0}$ and $-\sqrt{0}$ are both 0.

In some cases, it happens that there is no solution at all. For example, $x^2 = -1$ has no solution. If you square a number, it is always non-negative. Similarly, the following equation has no solution,

$$\begin{aligned} x^2 + 2x + 2 &= 0 \quad (28)\\ (x+1)^2 &= -1 \quad (29) \end{aligned}$$

since no number squared can be a negative number. In other words, if we make a quadratic equation into a form $(x+e)^2 = f$, and f turns out to be negative, there is no solution.

As an aside, in our later article "Complex numbers," we will introduce "imaginary numbers," defined by the condition that these numbers squared are negative. Then, we have solutions to the quadratic equations of the above type. However, these numbers are not "real," so we call them "imaginary." At present, just accept that there is no such ordinary (i.e. "real") number whose square is negative. Anyway, just as the mankind came up with the concept of negative number to subtract a bigger number from a smaller one, the mankind came up with the concept of imaginary number to find the square root of negative numbers. Imaginary numbers are essential as much as negative numbers are. For example, in our later article "Complex numbers" we will see the case where the use of imaginary number was necessary to find the *real* solution of cubic equation. (The cubic equation is an equation of the form $ax^3 + bx^2 + cx + d = 0$, where $a \neq 0$. If $a = 0$, and $b \neq 0$ it's just a quadratic equation.) If you are curious and eager to learn about imaginary numbers, you can jump into that article, and also read "Complex conjugate," the article following it.

Problem 1. Solve $x^2 = 2x + 3$. (Hint[1])

Problem 2. Solve $3x = x^2 + 2$. (Hint[2])

[1]This equation is equivalent to $x^2 - 2x - 3 = 0$.
[2]This equation is equivalent to $x^2 - 3x + 2 = 0$.

In solving quadratic equations, completing the square is sometimes too complicated such as in (22) or (24). For such cases, we can solve the following equation once and for all:

$$ax^2 + bx + c = 0 \tag{30}$$

and memorize its solution, and plug in appropriate a, b and c into this formula, whenever you have to solve a complicated quadratic equation. Let's solve this by assuming $a > 0$. Dividing (30) by a, we get

$$\begin{aligned}
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \qquad (31)\\
\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} &= 0\\
\left(x + \frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a}\\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a}\\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2}\\
x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{\sqrt{(2a)^2}} \qquad (32)\\
x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} \qquad (33)\\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (34)
\end{aligned}$$

Here, $D = b^2 - 4ac$ is called a "discriminant." If $D > 0$, we have two solutions. Namely, the two solutions are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{35}$$

If $D = 0$, we have one solution. Namely, we have

$$x = \frac{-b \pm 0}{2a} = -\frac{b}{2a} \tag{36}$$

as $+0$ and -0 are both just 0. If $D < 0$, we have no solution.

Problem 3. When we went from (32) to (33). Now, assume $a < 0$, re-derive the answer again, and confirm that the answer is again (34), or equivalently (35).

Problem 4. Solve the following equations. (Hint[3])

$$(4x + 3)(x - 3) = 2x - 16, \qquad (x + 3)(x - 2) = x^2 + 4x + 6$$

Problem 5. Suppose ϵ and λ are two solutions to (30). What is their sum $\epsilon + \lambda$? What is their product $\epsilon\lambda$?

[3] First, make them into the form (30).

You may be surprised that the answer to this problem comes out relatively simple compared to the calculation that involved. Is it a coincidence? No. There is a more elegant way to solve this problem.

To this end, we will show that if α is a solution to $x^2+ex+f=0$, it can be factorized as follows for some β:

$$x^2 + ex + f = (x - \alpha)(x - \beta) \tag{37}$$

To see this, first, notice that $x^2 + bx + c$ can be always expressed as follows

$$x^2 + ex + f = (x - \alpha)(x - \beta) + \gamma \tag{38}$$

for some β and γ. (If you divide $x^2 + ex + f$ by $(x - \alpha)$, then $(x - \beta)$ is the quotient and γ is the remainder.) Now, let's plug in $x = \alpha$ to both sides. The left-hand side is zero, because it is one of the solutions to $x^2 + ex + f = 0$, while the right-hand side is γ, because

$$(\alpha - \alpha)(\alpha - \beta) + \gamma = 0(\alpha - \beta) + \gamma = \gamma \tag{39}$$

So, we conclude $\gamma = 0$. This brings us back to (37). We also see that β is the other solution by plugging $x = \beta$ to both sides, which yield zero. Finally, we observe that (37) allows us to express e and f in terms of the solution α and β. By expanding, we get:

$$x^2 + ex + f = x^2 - (\alpha + \beta)x + \alpha\beta \tag{40}$$

Therefore, we conclude:

$$e = -(\alpha + \beta), \qquad f = \alpha\beta \tag{41}$$

In other words, the sum of the two solutions is $-e$ and the product of the two solutions is f.

To answer our calculation, let's convert $ax^2+bx+c = 0$ into the form $x^2+ex+f = 0$ by divding by a, i.e.,

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \tag{42}$$

In other words,

$$e = \frac{b}{a}, \qquad f = \frac{c}{a} \tag{43}$$

Thus, the sum of the two solutions is $-b/a$ and the product of the two solutions is c/a.

Problem 6. If the solutions to $x^2 + bx + c = 0$ are 2 and 3, what are b and c?

Problem 7. Solve $x^2 + 5x = 0$ by using (34).

Actually, there was no need to use (34), because $x^2 + 5x$ can be factored out as

$$x(x + 5) = 0 \tag{44}$$

If the product of two numbers is zero, one (or both) of them must be zero. This implies

$$x = 0 \text{ or } x + 5 = 0 \tag{45}$$

In other words,

$$x = 0, -5 \tag{46}$$

More generally, if a quadratic equation can be factored out as

$$(x - \alpha)(x - \beta) = 0 \tag{47}$$

either $(x - \alpha)$ must be zero, or $(x - \beta)$ must be zero, which implies

$$x = \alpha, \beta \tag{48}$$

Problem 8. If the solutions to $6x^2 + dx + e = 0$ are $\dfrac{1}{2}$ and $\dfrac{1}{3}$, what are d and e? (Hint[4])

Problem 9. Solve the following equations. (Hint[5])

$$\frac{1}{x-1} + \frac{1}{x-2} = \frac{3}{2}, \qquad \sqrt{x^2 - 3x} = x + 2 \tag{49}$$

Problem 10. Solve the following equations. (Hint[6])

$$x + y = 1 \tag{50}$$

$$x^2 = 4y/3 \tag{51}$$

Problem 11. If the following equation is an identity, what are the values for a, b, and c?

$$ax^2 + 3x + 4 = 2x^2 + bx + c \tag{52}$$

Problem 12. If the following equation is an identity, what are the values for a, b, and c?

$$x + 3 = ax^2 + bx(x+1) + c(x+1) \tag{53}$$

Problem 13. If the following equation is an identity, what are the values for a, b, and c?

$$\frac{x + \frac{1}{2}}{x(x+1)^2} = \frac{a}{x} + \frac{b}{x+1} + \frac{c}{(x+1)^2} \tag{54}$$

Problem 14. In this problem, we will solve the quadratic equation by an alternative method. Solve Problems 9-11 of "Polynomials, expansion and factoring" first, before solving this problem. In (40) and (41), we have seen that α, β, the solutions to $x^2 + bx + c = 0$ satisfy

$$\alpha + \beta = -b, \qquad \alpha\beta = c \tag{55}$$

[4] Notice $6(x^2 + \frac{d}{6}x + \frac{e}{6}) = 0$.

[5] For the first one, multiply both sides by $2(x-1)(x-2)$. For the second one, notice that the equation implies $(x^2 - 3x) = (x+2)^2$.

[6] Show first that the equations imply $x + 3x^2/4 = 1$.

Given this, assuming $\alpha \geq \beta$, obtain $\alpha^2 + \beta^2$ and $(\alpha - \beta)^2$ and $\alpha - \beta$ in terms of b and c. Finally, from $\alpha + \beta = -b$ and the value of $\alpha - \beta$ you got, obtain the expressions for α and β in terms of b and c. Finally, check if your answer is correct by plugging in $a = 1$ into (34).

Let me conclude this article with a comment. We can solve any quadratic equations using the method presented in this article. However, this trick doesn't work for other types of equations such as cubic or quartic equations. (Cubic equations are types of equations than can be represented in the form $ax^3 + bx^2 + cx + d = 0$, and quartic equations are types of equations that can be represented in the form $ax^4 + bx^3 + cx^2 + dx + e = 0$.) When I was still a young child, I read from a cartoon book that the solution to any cubic equation was found in the 16th century. So, I told my mom that I wanted to know the solution. She took me to the biggest library in Daejeon and found out a book that described this. It was before the time when the Internet was widely available and before the time when libraries used computers to aid in searching for a book. I also guessed that the solutions to cubic equations were important, since the solutions to quadratic equations are important, and cubic equations seemed to be the next least complicated equations. However, I came to learn much later that the solutions to the cubic equations are much less important in everyday lives or in physics or engineering than I imagined. At least, if the solutions to the cubic equations are ever needed, one can use a computer to get numerical solutions. (Numerical solutions are solutions in terms of pure values, such as $x = 2.05817\cdots$ as opposed to $x = \sqrt{2 + \sqrt{5}}$.) Therefore, most university students, regardless of their majors, do not learn the solution to the cubic equations even though it requires nothing beyond high school math.

The solution to quartic equations was also found in the 16th century, but the solutions to a quintic equation or higher degree of equations were never found. (Quintic equations are of the form $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$.) However, the 19th century French mathematician Évariste Galois proved that a general solution to such equations cannot exist. In other words, one cannot solve such equations using addition, subtraction, multiplication, division and roots only. (Of course, one can still obtain the solution numerically using computers.) The proof uses Galois theory, which math majors learn in "abstract algebra" class in their sophomore year. (In our essay "$\sqrt{2}$ as irrational number," we briefly mentioned that you can learn about "ring" and "field" in "abstract algebra" class.) At present, Galois theory is a branch of mathematics that finds no immediate applications in physics or engineering. Nevertheless, I read a Korean translation of a book on Galois theory aimed at laymen written by a Korean Japanese Korean-to-Japanese translator and novelist. The title roughly translates into English as "Galois theory which I teach to my thirteen-year old daughter." The author has absolutely no background in mathematics, but came upon a textbook for math majors in a bookstore in Seoul, deciphered it and wrote the book. Besides the concept of field, Galois theory uses the concept of group. In our later articles, we will explain what groups are, and what kind of an observation Galois made led to his proof that a general solution to a quntic equation doesn't exist.

Summary

- A type of equations that can be reduced to $ax^2 + bx + c = 0$ for $a \neq 0$, is called a "quadratic equation."
- A quadratic equation can be solved by completing the square.
- $(x+d)^2 = e$ has two real solutions if $e > 0$, one real solution $e = 0$, no real solution $e < 0$.
- If α and β are solutions to $x^2 + bx + c = 0$, we necessarily have

  $$x^2 + bx + c = (x - \alpha)(x - \beta)$$

- If $x^2 + bx + c$ can be factored out as

  $$x^2 + bx + c = (x - \alpha)(x - \beta)$$

  the solutions to $x^2 + bx + c = 0$ are α and β.
- The solution to $ax^2 + bx + c = 0$ where $a \neq 0$, is given by

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  $D = b^2 - 4ac$ is called a "discriminant." If $D > 0$ there are two solutions, if $D = 0$ there is one solution, if $D < 0$ there is no solution.