Quadratic inequalities and the Cauchy-Schwarz inequality

Having solved quadratic equations and learned about the graphs of quadratic polynomials, we can now solve quadratic inequalities. Suppose we want to solve the following inequality. We will assume $a > 0$, as we can consider the case $a < 0$ later.

$$ax^2 + bx + c > 0 \tag{1}$$

To solve this inequlaity, let's draw a graph $y = ax^2 + bx + c$ and $y = 0$. See the figures on the next page. There are three cases. In the first case, there is no solution to $ax^2 + bx + c = 0$, as the minimum of $ax^2 + bx + c$ is bigger than zero. In this case all x satisfy (1). From the last article, we know that the minimum is $-(b^2 - 4ac)/4a$. Thus, we have

$$-\frac{b^2 - 4ac}{4a} > 0 \tag{2}$$

which implies $b^2 - 4ac < 0$. In other words, the discriminant is negative as expected from that there is no solution to $ax^2 + bx + c = 0$.

In the second case, there is one solution to $ax^2 + bx + c = 0$. In other words, the minimum of ax^2+bx+c is zero. This corresponds to $b^2-4ac = 0$. Since $ax^2 + bx + c$ is always positive except when it is zero, the answer to (1) is given by all $x \neq \alpha$ where α is the (only) solution to $ax^2 + bx + c = 0$.

Problem 1. In this second case, using the fact that $x = \alpha$ is the only solution to $ax^2 + bx + c = 0$, express α in terms of a and b. (Hint[1])

In the third case, there are two solutions to $ax^2 + bx + c = 0$. If the smaller solution is α while the bigger solution is β, it is obvious from the graph that $ax^2 + bx + c$ is positive when $x < \alpha$ or $x > \beta$. This is the answer.

Summarizing when $a > 0$, the solution to (1) is

- If $b^2 - 4ac < 0$, all x.
- If $b^2 - 4ac = 0$, $x = -\dfrac{b}{2a}$
- If $b^2 - 4ac > 0$, $x > -\dfrac{b + \sqrt{b^2 - 4ac}}{2a}$ or $x < -\dfrac{b - \sqrt{b^2 - 4ac}}{2a}$

[1] Recall the position of vertex from the last article.

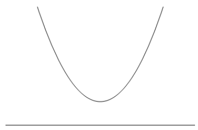

Figure 1: $b^2 - 4ac < 0$

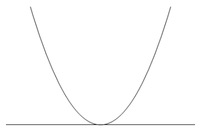

Figure 2: $b^2 - 4ac = 0$

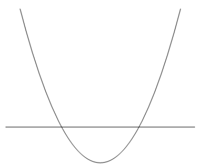

Figure 3: $b^2 - 4ac > 0$

Problem 2. Solve $ax^2 + bx + c \geq 0$ for $a > 0$. (Hint[2])

Problem 3. Solve $ax^2 + bx + c < 0$ for $a > 0$. (Hint[3])

Notice also that if $a < 0$ in (1), it can be converted to this type of inequality. For example, if we have to solve $-2x^2 + 3x + 4 > 0$, we can solve instead $0 > 2x^2 - 3x - 4$.

Problem 4. Notice that the following is satisfied for all t, as it is a sum of two non-negative numbers (if you square something you always get something non-negative):

$$(tX_1 + X_2)^2 + (tY_1 + Y_2)^2 \geq 0 \tag{3}$$

If you expand this, you get a degree-2 polynomial of t. By using the fact that its discriminant must be zero or negative, as it corresponds to the case of Fig. 1 and Fig. 2, show the following:

$$(X_1X_2 + Y_1Y_2)^2 \leq (X_1^2 + Y_1^2)(X_2^2 + Y_2^2) \tag{4}$$

This is known as the Cauchy-Schwartz inequality. More generally, the following is satisfied, if we consider $(tX_1+X_2)^2+(tY_1+Y_2)^2+(tZ_1+Z_2)^2+\cdots \geq 0$:

$$(X_1X_2+Y_1Y_2+Z_1Z_2+\cdots)^2 \leq (X_1^2+Y_1^2+Z_1^2+\cdots)(X_2^2+Y_2^2+Z_2^2+\cdots) \tag{5}$$

We will use this formula to prove the "triangle inequality" later.

Final comment. In our later article on quantum mechanics, you will learn Heisenberg's uncertainty principle. According to this principle, you cannot measure position and velocity of a particle both accurately at the

[2] Looking at the three figures again will be helpful.

[3] Except for when $ax^2 + bx + c > 0$ or $ax^2 + bx + c = 0$ is satisfied, this inequality is always satisfied. Either use this fact or think on your own by looking at the three figures closely again.

same time. The proof of Heisenberg's uncertainty principle uses the Cauchy-Schwarz inequality.

Problem 5. Solve $3x < x^2 + 2$. (Hint[4])

Problem 6. Solve $x^2 < 6x - 10$. (Hint[5])

Problem 7. Can the square of a certain number ever be smaller than the original number? At first glance, it may seem unlikely. If you square 2, it is 4 which is not smaller than 2. If you square 100, it is 10000 which is not smaller than 100. The situation does not change for negative numbers. If you square -3 it is 9, which is not smaller than -3. However, show that the square of a certain number *can be smaller* than the original number, by solving $x^2 < x$. After obtaining the solution, check yourself with an explicit example that the square can be smaller than the original number.

Summary

- The Cauchy-Schwartz inequality can be obtained by considering the discriminant of a quadratic inequality and is given by

$$(X_1X_2 + Y_1Y_2)^2 \leq (X_1^2 + Y_1^2)(X_2^2 + Y_2^2)$$

[4]Solve $x^2 - 3x + 2 > 0$

[5]Solve $x^2 - 6x + 10 < 0$