Root, cube root and n^{th} root, revisited

In an earlier article, we introduced the concept of root, cube root and nth root. In this article, we will delve into their properties.

First,

$$\sqrt{x}\sqrt{y} = \sqrt{xy} \tag{1}$$

One can show this as follows:

$$(\sqrt{x}\sqrt{y})^2 = (\sqrt{x})^2(\sqrt{y})^2 = xy \tag{2}$$

Similarly, one can show

$$\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}} \tag{3}$$

Using these relations, we can play around with roots. For example,

$$\sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}, \qquad \sqrt{18} = \sqrt{9}\sqrt{2} = 3\sqrt{2} \tag{4}$$

Problem 1. Show the following. (Hint[1])

$$\sqrt{x}\sqrt{y}\sqrt{z} = \sqrt{xyz} \tag{5}$$

Problem 2. Show the following.

$$\sqrt[3]{x}\,\sqrt[3]{y} = \sqrt[3]{xy} \tag{6}$$

Similarly, we have

$$\sqrt[n]{x}\,\sqrt[n]{y} = \sqrt[n]{xy} \tag{7}$$

Finally, let us mention how the concepts of root, cube root and n^{th} root connects to exponents. First, notice that expressions such as $x^{1/2}$ wouldn't make much sense at first glance, since you cannot multiply a number "half" times. However, there is a way to assign a value to this expression in a consistent way. (Remember, we assign values to cases in which a number is multiplied "0" times and "negative" times. There is nothing we can't do.) So, let's see. Observe:

$$(x^{\frac{1}{2}})^2 = x^{\frac{1}{2}\cdot 2} = x^1 = x \tag{8}$$

[1] Show $(\sqrt{x}\sqrt{y}\sqrt{z})^2 = xyz$.

In other words, the square of $x^{1/2}$ is x. Therefore, we conclude

$$x^{\frac{1}{2}} = \sqrt{x} \tag{9}$$

Problem 3. Show that the cube root of x is $x^{\frac{1}{3}}$. (Hint[2])
You just showed

$$x^{\frac{1}{3}} = \sqrt[3]{x} \tag{10}$$

Similarly, you can also show that

$$, \qquad x^{\frac{1}{n}} = \sqrt[n]{x} \tag{11}$$

Given this, what would expressions like $x^{2/3}$ mean? We have:

$$x^{\frac{2}{3}} = (x^{\frac{1}{3}})^2 = (\sqrt[3]{x})^2 = \sqrt[3]{x^2} \tag{12}$$

where in the last step we used the result of Problem 2 (with y replaced by x). There is another way to derive this result:

$$x^{\frac{2}{3}} = (x^2)^{\frac{1}{3}} = \sqrt[3]{x^2} \tag{13}$$

More generally,

$$x^{\frac{p}{q}} = (\sqrt[q]{x})^p = \sqrt[q]{x^p} \tag{14}$$

Problem 4. Simplify or evaluate the following. (Hint[3])

$$8^{\frac{4}{3}} =?, \qquad \sqrt{12}\sqrt{3} =?, \qquad \sqrt{18} - \sqrt{8} =?, \qquad 4^{-\frac{3}{2}} =?, \qquad \left(\frac{1}{4}\right)^{-\frac{1}{2}} =?$$

Problem 5. Simplify the following. Assume that b is positive. (Hint[4])

$$\left(\frac{\sqrt{ab}}{b}\right)^2 =?, \qquad \frac{a}{b}\sqrt{\frac{b}{c}} =?, \qquad \frac{a^{3/2}}{ab} =?, \qquad \sqrt{\frac{a^4}{b^4}} =? \tag{15}$$

Problem 6. Solve the following equations.

$$\sqrt{x} = 3 \tag{16}$$

$$\sqrt{x+1} = 0 \tag{17}$$

$$\sqrt[3]{x} = 3 \tag{18}$$

$$\sqrt{2x-3} = 2 \tag{19}$$

$$\sqrt{\sqrt{x}} = 2 \tag{20}$$

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[2] Show $(x^{\frac{1}{3}})^3 = x$.
[3] $\sqrt{18} = \sqrt{9}\sqrt{2}$, $\sqrt{8} = \sqrt{4}\sqrt{2}$.
[4] $\frac{a}{b} = \frac{a}{\sqrt{b}\sqrt{b}}$

Problem 7. Solve the following equations. (Hint[5])

$$\sqrt{x} + 4 = 2\sqrt{x} + 1 \tag{21}$$

$$\sqrt{x+2} + 1 = 3\sqrt{x+2} - 1 \tag{22}$$

Problem 8. Explain why the following equations have no solutions.

$$\sqrt{x} = -4 \tag{23}$$

$$\sqrt{x} + 4 = -\sqrt{x} \tag{24}$$

$$x^2 + 5 = 3 \tag{25}$$

Problem 9. Solve the following equations. (Hint[6])

$$x^2 + 4 = 9 \tag{26}$$

$$x^2 + 3 = 3x^2 \tag{27}$$

$$\sqrt{1 - x^2} = x \tag{28}$$

Summary

- $\sqrt{x}\sqrt{y} = \sqrt{xy}$
- $\dfrac{\sqrt{x}}{\sqrt{y}} = \sqrt{\dfrac{x}{y}}$
- $\sqrt[3]{x}\,\sqrt[3]{y} = \sqrt[3]{xy}$
- $x^{\frac{1}{2}} = \sqrt{x}$
- $x^{\frac{1}{n}} = \sqrt[n]{x}$
- $x^{\frac{p}{q}} = (\sqrt[q]{x})^p = \sqrt[q]{x^p}$

[5] For the first one, obtain the value for $\sqrt{x}$ first. For the second one, obtain the value for $\sqrt{x+2}$ first.

[6] The last equation implies $1 - x^2 = x^2$.