Root, cube root and n^{th} root

Let's say you squared a non-negative number and got 9. What is the original number? The answer is 3, as

$$3^2 = 3 \times 3 = 9 \tag{1}$$

Let's say you squared a non-negative number and got 4. What is the original number? The answer is 2, as

$$2^2 = 2 \times 2 = 4 \tag{2}$$

Let's say you squared a non-negative number and got 49. What is the original number? The answer is 7, as

$$7^2 = 7 \times 7 = 49 \tag{3}$$

Let's say you squared a non-negative number and got 25. What is the original number? The answer is 5, as

$$5^2 = 5 \times 5 = 25 \tag{4}$$

Let's say you squared a non-negative number and got 11. What is the original number? We call this number $\sqrt{11}$. We pronounce $\sqrt{11}$ as "root 11" or "square root of 11." How big is this number? As 3 squared is 9 and 4 squared is 16, this number must be somewhere between 3 and 4. In other words,

$$3 < \sqrt{11} < 4 \tag{5}$$

Now notice further

$$3.3^2 = 10.89 < 11 < 3.5^2 = 12.25 \tag{6}$$

So, we see that $\sqrt{11}$ is between 3.3 and 3.5. By continuing this process, we can make the range as small as possible to find more accurate value for $\sqrt{11}$. However, actually, there is no need to perform such a tedious work. You can just use a calculator to find $\sqrt{11}$. For example, if you google "square root 11," you will get

$$\sqrt{11} \approx 3.31662479036 \tag{7}$$

Certainly, by using a better software, you can obtain more exact value, i.e., more digits for $\sqrt{11}$. Of course, sometimes we can get the exact value for a square root without using a calculator. In our case of (1), (2), (3), and (4), we have

$$\sqrt{9} = 3, \qquad \sqrt{4} = 2, \qquad \sqrt{49} = 7, \qquad \sqrt{25} = 5 \tag{8}$$

More formally, if x is a non-negative number that satisfies $x^2 = y$, we have:

$$x = \sqrt{y} \tag{9}$$

So, why do we need the condition that x is a non-negative number? Let's see what happens when we remove this condition. If x is a number that satisfies $x^2 = 25$, what is x? There are two possibilities: x can be either 5 or -5 as

$$5 \times 5 = 25 \tag{10}$$

$$(-5) \times (-5) = 25 \tag{11}$$

Therefore, we see that we can get two answers, if we remove the condition that the answer must be non-negative. i.e., if we allow a negative answer. This is problematic, because we would not know whether $\sqrt{25}$ means 5 or -5.

Problem 1. Evaluate the following:

$$\sqrt{0}, \qquad \sqrt{1}, \qquad \sqrt{100}, \qquad \sqrt{81}, \qquad \sqrt{\frac{1}{4}}$$

These examples were carefully chosen so that you don't need to use a calculator to find the answer.

Problem 2. Using a calculator, evaluate the following up to three decimal digits:

$$\sqrt{3}, \qquad \sqrt{5}$$

Now, let me introduce the cube root. If $x^3 = y$, then $x = \sqrt[3]{y}$. For example, as $2^3 = 8$, we have $2 = \sqrt[3]{8}$. Notice that this is *not* equal to $3\sqrt{8}$, which means 3 multiplied by $\sqrt{8}$. The 3 in a cube root is written in a small size.

There is also another big difference between the root and the cube root. Notice that the root of a negative number doesn't exist; if you multiply a number by itself you get always a non-negative number. (Remember if you multiply -3 by -3 you get 9 not -9.) On the other hand, the cube root of a negative number exists. For example, $\sqrt[3]{-27} = -3$ as $(-3)^3 = (-3) \times (-3) \times (-3) = -27$.

We can actually generalize the square root and the cube root to n^{th} root. For example, if $x^n = y$ is satisfied, we have $x = \sqrt[n]{y}$.

Problem 3. Evaluate the following.

$$\sqrt[3]{27} =? \qquad \sqrt[5]{-1} =? \qquad 3\sqrt{16} =? \qquad \sqrt{3\sqrt{4}+3} =?$$

Now, notice

$$(3\sqrt{2})^2 = 3\sqrt{2} \times 3\sqrt{2} = 3 \times 3 \times (\sqrt{2})^2 = 18 \tag{12}$$

Problem 4. Evaluate the following.

$$(2\sqrt{3})^2, \qquad (4\sqrt{2})^2, \qquad (\sqrt{2})^4, \qquad (\sqrt{3.4561})^2$$

Problem 5. From (12), explain why $\sqrt{18}$ is $3\sqrt{2}$.
Problem 6. Evaluate the following.

$$\sqrt{4^2} =?, \qquad \sqrt{(-4)^2} =?, \qquad \sqrt{6^2} =? \qquad \sqrt{(-6)^2}$$

If you solved the last problem correctly, you would agree that

$$\sqrt{(-5)^2} = \sqrt{5^2} = 5 \tag{13}$$

In other words, for a non-negative number a, we have

$$\sqrt{a^2} = a = |a| \tag{14}$$

as a and $|a|$ are the same if a is non-negative. If a is negative, we have

$$\sqrt{a^2} = |a| \tag{15}$$

For example,

$$\sqrt{(-5)^2} = 5 \tag{16}$$

which is equal to $|-5|$.

As a is always either non-negative or negative, from (14) and (15), we conclude

$$\sqrt{a^2} = |a| \tag{17}$$

for any number a.

Problem 7. Explain why we always have

$$\sqrt{b^2} = -b \tag{18}$$

if b is a negative number. If you are not sure how to explain this, it is sufficient to give an example.

Summary

- $x = \sqrt{y}$ is a non-negative number that satisfies $x^2 = y$.
- $x = \sqrt[n]{y}$ satisfies $x^n = y$.
- $\sqrt{x^2} = |x|$.