Set theory

1 Set, elements and the empty set

According to Wikipedia, a set is a well-defined collection of distinct objects. For example, the set of natural numbers smaller than 6 is a well-defined set. One can express this set by two different methods. First, by explicitly writing. If we call this set "A," we have

$$A = \{1, 2, 3, 4, 5\} \tag{1}$$

Second, by writing out the conditions that their members belong to this set. Namely,

$$A = \{x|x \text{ is a natural number and } x < 6\} \tag{2}$$

Of course, x here, doesn't have a specific meaning. We could have as well written

$$A = \{y|y \text{ is a natural number and } y < 6\} \tag{3}$$

The objects that belong to a set are called "elements." It is perfectly OK that elements of a set are not numbers. For example, the set of male students who are taking Mr. Mayerhofer's German class on Monday and Wednesday evenings is a perfectly good set. It is

$$B = \{\text{Kyuhyon}, \text{Kyungtae}, \text{Youngsub}\} \tag{4}$$

If an object is an element of a set, we use the notation "$\in$." For example,

$$2 \in A, \qquad \text{Kyungtae} \in B \tag{5}$$

If an object is not an element of a set, we use the notation "$\notin$." For example,

$$5 \notin A, \qquad \text{Ryan} \notin B \tag{6}$$

It is also possible that a set contains no element. For example, the set of students who are taking Youngsub's French class is given by

$$C = \{\} \tag{7}$$

We call such a set "the empty set" and denote it as $\emptyset$. For example,

$$C = \emptyset \tag{8}$$

The number of elements of a set A is often denoted by $|A|$. For example, in our case, $|A| = 5$, $|B| = 3$, $|C| = 0$.

2 Subset

Let's say that every element of a set D is also an element of a set E. Then, we say "D is a subset of E," and denote this fact as

$$D \subset E \tag{9}$$

For example, if D is a set of positive odd numbers smaller than 5, we have

$$D = \{1, 3\} \tag{10}$$

and we know that 1 and 3 are also elements of A. Thus, we see D is a subset of A. In other words, $D \subset A$. For a graphical representation, see Fig. 1.

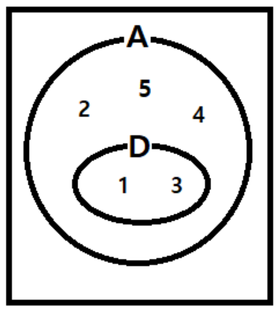

Figure 1: $D \subset A$

Let's say that F is a set of positive odd numbers smaller than 4. Then, we have

$$F = \{1, 3\} \tag{11}$$

Now, notice that every element of F is also an element of D. Therefore, we can write $F \subset D$. Similarly, it is easy to check that every element of D is also an element of F. Thus, we can write $D \subset F$. We also notice that F is the same set as D. In other words, $D = F$. More generally, we have

$$\text{If } G \subset H \text{ and } H \subset G, \text{ then } G = H \tag{12}$$

If I is a subset of J, but I is not equal to J, we say I is a proper subset of J. In other words, every element of I is an element of J (i.e., J includes I or $I \subset J$), but there is at least one element in J that is not an element of I. In our examples, D is a proper subset of A, but not a proper subset of F, even though it is a subset of F.

3 Unions, intersections and set difference

The union of K and L, denoted by $K \cup L$, is the set of objects that are elements of either K or L. For example, if $A = \{1, 2, 3, 4\}$ as in (1) and M is the set of positive even numbers smaller than 9, i.e.,

$$M = \{2, 4, 6, 8\} \tag{13}$$

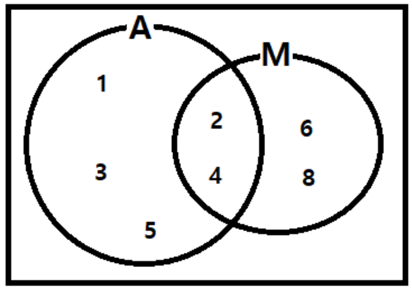

Figure 2: $A \cap M = \{2, 4\}, \qquad A \setminus M = \{1, 3, 5\}$

we have

$$A \cup M = \{1, 2, 3, 4, 5, 6, 8\} \tag{14}$$

The intersection of K and L, denoted by $K \cap L$ is the set of objects taht are elements of both K and L. For example, we have

$$A \cap M = \{2, 4\} \tag{15}$$

See Fig. 2. You see that both 2 and 4 are included in A and M.

Diagrams like Fig. 1 and Fig. 2, which show the relation between sets are called "Venn diagrams."

Set difference of K and L, denoted as $K \setminus L$ is a set of elements that belong to K but not L. For example, if you see Fig. 2. yo see that 1, 3, 5 are elements of A, but not elements of M. Thus,

$$A \setminus M = \{1, 3, 5\} \tag{16}$$

Problem 1. What is $M \setminus A$?

4 $|S \cup K| = |S| + |K| - |S \cap K|$

Now, let's see a concrete application of what we have learned, especially Venn diagram. Let's say the number of students who passed Spanish exam is $40 (= |S|)$, the number of students who passed Korean exam is $50 (= |K|)$, and the number of students who passed both the Spanish exam and the Korean exam is $8 (= |S \cap K|)$. Then, how many students passed either the Spanish exam or the Korean exam? In other words, what is $|S \cup K|$?

See Fig. 3. The students who passed Spanish exam (40 students) are in the blue region and the green region, and the students who passed Korean exam (50 students) are in the yellow region and the green region. In particular, the students who passed both exams (8 students) are in the green region. In other words,

$$|S| = \text{Blue} + \text{Green} = 40 \tag{17}$$

$$|K| = \text{Yellow} + \text{Green} = 50 \tag{18}$$

$$|S \cap K| = \text{Green} = 8 \tag{19}$$

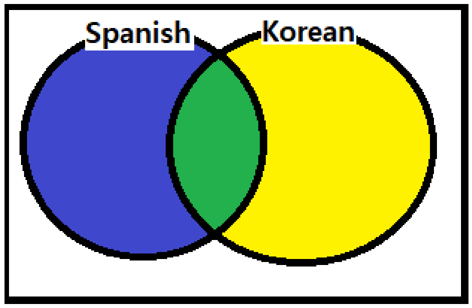

Figure 3: Spanish exam and Korean exam

By plugging (19) to (17), we get Blue $= 32$, and by plugging (19) to (18), we get Green $= 42$. What we want to calculate is the total number of students in blue, green, yellow regions. Thus, the answer is

$$|S \cup K| = \text{Blue} + \text{Green} + \text{Yellow} = 32 + 8 + 42 = 82 \tag{20}$$

There is a slightly easier way to solve this problem. By summing (17) and (18), we get

$$|S| + |K| = \text{Blue} + 2\ \text{Green} + \text{Yellow} \tag{21}$$

If we subtract (19) from this expression, we get

$$|S| + |K| - |S \cap K| = \text{Blue} + \text{Green} + \text{Yellow} \tag{22}$$

However, notice that this is exactly the same expression as (20). Thus, we get

$$|S \cup K| = |S| + |K| - |S \cap K| = 40 + 50 - 8 = 82 \tag{23}$$

In other words, if we simply add the number of students who passed Spanish exam and the number of students who passed Korean exam, to calculate the number of students who passed either exams, we are double counting the number of students who passed both exams. (See the factor 2 in front of "Green" i.e., $|S \cap K|$, in(21).) Therefore, we have to subtract the number of students who passed both exams, i.e., $|S \cap K|$.

Problem 2. Which region corresponds to $S \setminus K$? Blue, green, yellow? How about $K \setminus S$?

Problem 3. Answer whether each of the following statement is true. Drawing Venn diagrams may be helpful. (We assume that both the number of elements in A and the number of elements in B are finite. For example, neither A nor B can be a set of natural numbers which has infinite elements.[1])

[1]Strange things happen when a set has infinite elements. For example, the set of even number is a proper subset of the set of integer. In other words, there are integers that are not even numbers (they are odd numbers). However, it turns out that the number of even numbers and the number of integers are the same. Unfortunately, it is beyond scope of this article to discuss why it is so.

- If $A \subset B$ and $B \subset A$, $|A| = |B|$.
- If $A \subset B$, $|A| < |B|$.
- If $A \subset B$ and $B \subset C$, $A \subset C$.
- $(A \setminus B) \cup B = A \cup B$.
- $(A \setminus B) \subset A$.
- $A \setminus B$ is a proper subset of A.
- If $A \setminus B = \emptyset$, $A = B$.
- $(A \setminus B) \cup (B \setminus A) = A \cup B$.
- $(A \cap B) \cap C = A \cap (B \cap C)$.

Summary

- A set is a well-defined collection of distinct objects.
- The objects that belong to a set are called "elements."
- If A is an element of set B, we write it as $A \in B$.
- If A is not an element of set B, we write it as $A \notin B$.
- The number of elements of set A is denoted by $|A|$.
- If all the elements of A are the elements of B, we write it as $A \subset B$. In other words, B includes A.
- If $A \subset B$ and $B \subset A$, then $A = B$.
- If $A \subset B$ but $A \neq B$, we say A is a proper subset of B. In other words, A is a subset of B, but there is at least one element in B that is not an element of A.
- The set of elements, which are elements of both A and B is denoted by $A \cap B$.
- The set of elements, which are elements of either A or B is denoted by $A \cup B$.
- The set of elements, which are elements of A, but not B is denoted by $A \setminus B$.
- $|A \cup B| = |A| + |B| - |A \cap B|$.