System of linear equations, part II: three or more unknowns

In the last article, we solved equations with two unknowns. In this article, we explain how to solve equations with three or more unknowns.

Remember how we solved equations with two unknowns in the last article. We had two equations with two unknowns, and manipulated them to turn them into one equation with one unknown by eliminating one of the variables. Equations with three unknowns are similar. We have three equations, and by eliminating one of the three variables, we can turn them into two equations with two unknowns. Since we know how to solve two equations with two unknowns, the job is done. Now you can easily guess how to solve equations with four unknowns. We have four equations with four unknowns, and by eliminating one of the four variables, we can turn them into three equations with three unknowns, and so on.

Of course, without examples, this sounds somewhat abstract. So, let me give you an example.

$$\begin{aligned} x + 2y + z &= 4 \qquad (1) \\ x - 2y - 3z &= 0 \qquad (2) \\ 2x - y - z &= 1 \qquad (3) \end{aligned}$$

Subtracting (2) from (1), we get:

$$4y + 4z = 4 \qquad (4)$$

Multiplying (1) by 2, we get:

$$2x + 4y + 2z = 8 \qquad (5)$$

Subtracting (3) from the above equation, we get:

$$5y + 3z = 7 \qquad (6)$$

Now, we know how to solve (4) and (6). Multiplying (4) by 5/4, we get:

$$5y + 5z = 5 \qquad (7)$$

Subtracting (6) from the above equation, we get $2z = -2$. Therefore, we conclude $z = -1$. Then, by plugging this back to (4) or (6), we get $y = 2$. Then, by plugging these back into (1) or (2) or (3), we get $x = 1$.

We solved this problem by eliminating x first, but we could have solved it by eliminating y or z first. Of course, we are guaranteed to get the same answer.

Let me conclude this article with some comments. If there is one unknown, we need one equation. If we had zero equations, we could not determine the unknown. If we had two equations, we have more equations than needed. In other words, one equation is redundant. If there are two unknowns, we need two equations. If we had one equation, we cannot completely determine the unknowns: we only have one equation that relates these two unknowns, so there are infinitely many solutions; any set of two numbers that satisfy the one equation are solutions. If we had three equations, we have more equations than needed; we have one extra equation. If solutions exist, this extra equation is redundant, since we can already determine the solutions without the extra equation. Otherwise, there will be no solution at all, if the solution obtained from the first two equations doesn't satisfy the third equation. For example, if we solve the following equations

$$\begin{aligned} x + y &= 3 \\ x - y &= 1 \\ 2x + y &= 5 \end{aligned} \tag{8}$$

we already find $x = 2$, $y = 1$ from the first two equations, and the third equation is trivially satisfied as $2 \cdot 2 + 1 = 5$. On the other hand, if we solve the following equations

$$\begin{aligned} 2x + y &= 3 \\ x + y &= 2 \\ 2x - y &= 0 \end{aligned} \tag{9}$$

we find $x = 1$, $y = 1$ from the first two equations, but if we plug in these values to the third equation, it is not satisfied, so there are no solutions at all.

Generalizing, if we have n unknowns, we need exactly n "linearly independent" equations. (I will explain what "independent" means soon.) If we had less than n equations we can never completely determine the unknowns, although we would know some relations among them. In this case, we say the unknowns are under-determined. If we had more than n equations, some equations are necessarily redundant (on the conditions that the solutions exist at all), and the unknowns are over-determined. Otherwise, the solutions do not exist.

However, it turns out that you cannot completely determine the unknowns even when you have n equations for n unknowns, if the equations are not "linearly independent" but "linearly dependent." These are very special cases. For example, let's say

$$\begin{aligned} x + 2y - 3z &= 4 \\ 2x - y + z &= -3 \\ 3x + y - 2z &= 1 \end{aligned} \tag{10}$$

If you sum the first equation and the second equation you get the third equation. So, the third equation doesn't give an extra condition, and we effectively have two equations. The unknowns are under-determined, as the equations are not "linearly independent" but "linearly dependent" on one another. "Linear dependence" means that one of the equations is expressible in terms of linear combinations of others (i.e. multiplying certain numbers to other equations and adding them up) which make the former redundant. Here, we gave you an example in which there are 3 unknowns and 3 equations, but one can easily imagine that the business gets complicated if the number of the unknowns (i.e. the number of equations) is big. Actually there is a systematic way to check whether equations are linearly independent. This is an important concept in mathematics.

In our later article "Row reduction and echelon form," we will mathematically prove that we need exactly n linearly independent equations if we have n unknowns. Also, in our later article "Linear independence, linear dependence and basis," we will mathematically define linear independence and linear dependence.

Problem 1. Solve the following equations.

$$x + y + z = 0 \tag{11}$$
$$2x + 3y = 1 - 3z \tag{12}$$
$$3x - 2z = -5 \tag{13}$$

Problem 2. Show that there is no solution to the following equations (Hint[1]):

$$x + 2y - 3z = 4 \tag{14}$$
$$2x - y + z = -3 \tag{15}$$
$$3x + y - 2z = 0 \tag{16}$$

Summary

- If you have a system of linear equations with n unknowns, you can solve it by first getting rid of one of the unknowns and make it a system of linear equations with $n-1$ unknowns. You can repeat this process to finally get a linear equation with 1 unknown, which you know how to solve. Then, you can plug it back to the equations one by one to get all the unknowns.

[1] Sum the first two equations and compare the result with the third.