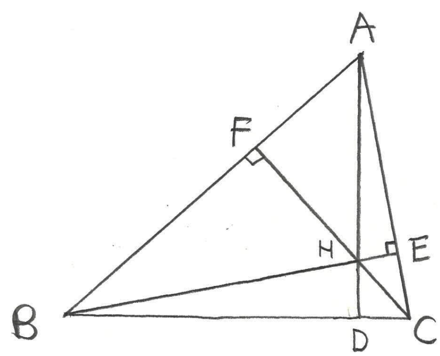
The three altitudes of a triangle always meet at a point

In an earlier essay, we mentioned that 12-year old Albert Einstein was impressed by the fact that the three altitudes of a triangle always meet at a point, and this can be proven with certainty. In this article, you will be invited to prove by two different methods. I will provide the steps. See the triangle below.



1 Geometric method

In the figure, $\overline{AB}$ and $\overline{CF}$ are perpendicular, and $\overline{BE}$ and $\overline{AC}$ are perpendicular as well. $\overline{BE}$ and $\overline{CF}$ meet at the point H. If you extend the line from A to H, it meets with $\overline{BC}$ at the point D. If we prove that $\overline{BC}$ and $\overline{AD}$ are perpendicular, our job is done.

Problem 1. Explain why $\square AFHE$ lies in a circle. Thus, we obtain $\angle HAE = \angle HFE$.

Problem 2. Explain why $\square FECB$ lies in a circle. Thus, we obtain $\angle HFE = \angle HBC$.

Combining the result of Problem 1. and Problem 2. we have $\angle HAE = \angle HBC$. Thus, we obtain $\square AEDB$ lies in a circle.

Problem 3. From this fact, conclude that $\overline{AD}$ and $\overline{BC}$ are perpendicular to each other.

2 Algebraic method

This time, we will assume that $\overline{AD}$ and $\overline{BC}$ are perpendicular. Then, we will calculate the coordinate of the point $\overline{CF}$ meets with $\overline{AD}$. And, we will also calculate the coordiante of the point $\overline{BE}$ meets wit $\overline{AD}$. If these two points are located at the same place, then, our proof is done.

As in the last article, a clever choice of coordinates for A, B, C make the calculation much easier. We will assume that D is located at (0,0), so that the x-coordinate of the two points mentioned are both 0, which make the calculation simpler. We will also assume that B and C are on the x-axis. Given this, we will denote the coordinates of the three vertices as follows:

$$A = (0, a), \qquad B = (b, 0), \qquad C = (c, 0) \tag{1}$$

Problem 4. Find the slope of $\overline{AB}$.

Problem 5. Find the slope of $\overline{CF}$.

Problem 6. Find the equation of the graph for $\overline{CF}$

Problem 7. Obtain the point where $\overline{CF}$ meets $\overline{AD}$ by plugging $x = 0$ to the answer to Problem 6.

Problem 8. Obtain the point where $\overline{BE}$ meets $\overline{AC}$ by exchanging b and c in the answer to Problem 7. Thus, check that the two points are located at the same position.

Summary

- The three altitudes of a triangle always meet at a point.