The triangle inequality

If you have correctly solved Problem 3 of the last article, you must have figured out that you cannot draw a triangle with sides 2, 3 and 6. Nor can you draw a triangle with sides 3, 4 and 7. Let's see again what happened. See Fig. 1 for the attempt to draw a triangle with sides 2, 3 and 6.

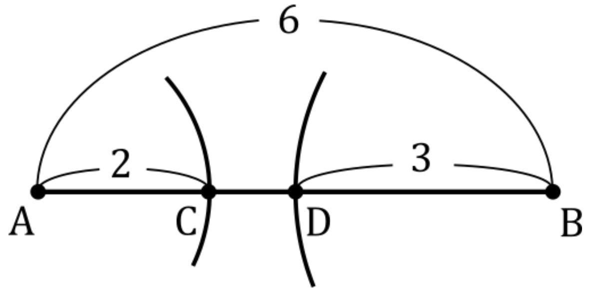

Figure 1: Attempt to draw a triangle with sides 2, 3 and 6.

You see that the two circles with radii 2 and 3 never meet each other, as their centers are located apart too far away, namely by 6.

Notice that this is because

$$6 > 2 + 3 \tag{1}$$

This is the reason why we cannot draw a triangle with sides 2, 3 and 6.

Let's check the other example in Problem 3 of the last article. Let's draw a triangle with sides 3, 4 and 7. See Fig. 2.

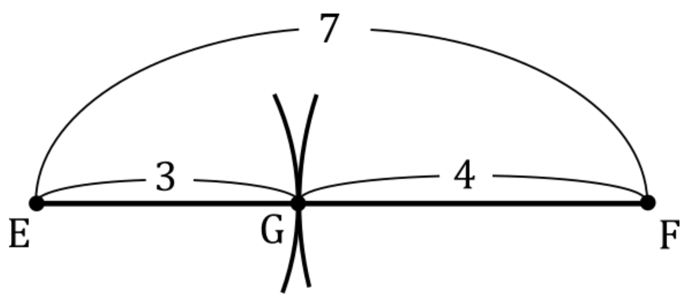

Figure 2: Attempt to draw a triangle with sides 3, 4 and 7.

The two circles indeed meet at the point G, but G lies in $\overline{EF}$, one of the sides of "could-have-been" triangle. "Triangle" EFG is not a triangle, because the sides $\overline{EG}$ and $\overline{FG}$ lie in another side $\overline{EF}$; a triangle must have three distinctive line segments. The problem is that the two circles met at a point on $\overline{EF}$. If they met elsewhere, we would surely have been able to draw the triangle.

Notice that the two circles meet at the point on $\overline{EF}$, because the equality

$$7 = 3 + 4 \tag{2}$$

holds.

Let's summarize what we have just learned, by denoting the three sides of triangle by a, b and c. From (1), we see that we cannot draw a triangle, if

$$a > b + c \tag{3}$$

or, from (2),

$$a = b + c \tag{4}$$

In other words, we cannot draw a triangle, if

$$a \geq b + c \tag{5}$$

So, only when (5) is not satisfied, can we have the chance to draw a triangle. This is when

$$a < b + c \tag{6}$$

To draw a triangle, this condition must be satisfied for all the three sides of the triangle, i.e., $b < a + c$, and $c < a + b$, i.e., the length of one side must be smaller than the sum of the lengths of the other two sides. In reality, we don't need to check this three times; checking that the longest side is smaller than the sum of the other two sides is enough. For example, if we have 4, 5 and 7 as the sides of triangle, checking $7 < 4 + 5$ is enough, because $4 < 7 + 5$ is automatically satisfied because $4 < 7$, and $5 < 7 + 4$ is automatically satisfied because $5 < 7$; if the side is not the longest one, the sum of the length of the other two sides is automatically bigger, as the length of the longest side alone is already bigger than the length of the original side we are comparing.

Let's summarize what we have found: if (5) is satisfied for any sides, we cannot draw the triangle. What is its contrapositive? If we have drawn a triangle, it must satisfy (6). In other words, a triangle always satisfies (6), Thus, (6) is called the "triangle inequality."

Let me give you another reasoning why the triangle inequality must be correct. To this end, let me explain what a "straight line" is. See Fig. 3.

A straight line JK is the shortest path between the point J and K. There are infinitely many possible paths between J and K, and the shortest

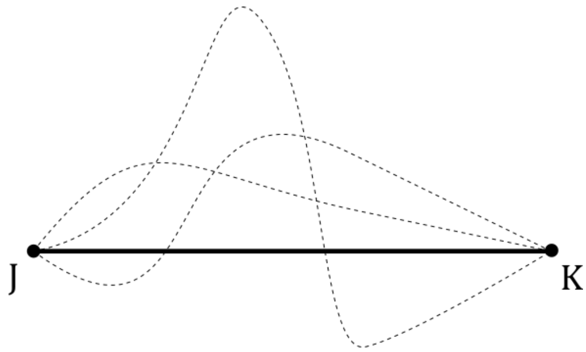

Figure 3: Straight and non-straight lines connecting points J and K.

path among them is the straight line. In the figure, three possible paths that are *not* the shortest path are denoted by the dotted lines, and the shortest path, the straight line, is denoted by the solid line.

This definition of a straight line may sound weird, but that is how mathematicians define a straight line.

Given this, note that a triangle has three vertices, say, J, K and L. These three vertices must not be colinear, i.e., must not lie in one straight line. Otherwise, we will have a stiuation like Fig. 2, and that won't be a triangle. The three sides of the triangle are the three straight lines, each of which connect two of the three vertices. Consider the side $\overline{JK}$. See Fig. 4.

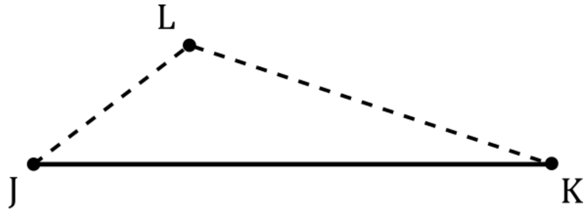

Figure 4: Straight lines connecting points J and K, but passing by point L first is longer than the straight line connecting points J and K directly.

It is the shortest path between J and K. So, a path that starts from J and pass L and ends at K is longer than $\overline{JK}$, as long as L does not lie on $\overline{JK}$. Indeed, L does not lie on $\overline{JK}$, because the triangle JKL would not have been a triangle otherwise. Notice that the path that we just mentioned has the length $\overline{JL} + \overline{LK}$.[1] Therefore, we conlcude

$$\overline{JK} < \overline{JL} + \overline{LK} \tag{7}$$

which is exactly the triangle inequality (6).

[1] Of course, only if the path from J to L and the path from L to K are straight lines.

In our later article "An algebraic proof of the triangle inequality," we will give you yet another proof for the triangle inequality. The algebraic approach in that article will be completely different from the approach we have taken in this article. In particular, you will see that Cauchy-Schwarz inequality you learned earlier will be crucial to this proof.

Finally, we want to mention that not only (6) is a necessary condition to draw the triangle, but also it is a sufficient condition. In other words, if I give you three positive numbers, the biggest of which is smaller than the sum of the other two, you can always draw a triangle with sides given by these three numbers. If you are not sure, think about why.

Summary

- In a triangle, the length of any one side is always smaller than the sum of the lengths of the other two sides.