Symbols and expressions

Let's say Steve ate 4 apples yesterday and 2 apples today. How many total apples has he eaten? You know the answer. 4+2=6. 6 apples.

Now, let's say Mary ate apples yesterday, but you don't know how many. But, you know, she ate 3 apples today. How many total apples has she eaten? We don't know, because we don't know how many apples she ate yesterday. But what we know for sure is that during these two days she has eaten 3 more apples than she had eaten yesterday. Would there be a simple way to express this?

To this end, let's say that she ate x apples yesterday. Then, she has eaten total $x+3$ apples. This is how we express unknowns using symbols such as x.

There is no reason that there can be only one symbol. There can be two symbols or more. For example, if Sylvia ate x apples yesterday and y apples today, she has eaten a total of $x+y$ apples. Similarly, if Larry ate x apples every day for seven days, he ate a total of $7x$ apples. You see here that I have omitted the multiplication sign; instead of $7 \times x$ apples, I have simply written $7x$ apples. This is a convention. Let me give you another example, if Josh ate a apples every day for b days, he has eaten a total of ab apples. Notice that the $\times$ sign is missing by convention.

We can actually manipulate and treat symbols such as x and y as if they are ordinary numbers; they satisfy the commutative, associative and distributive properties. For example, as these symbols satisfy the commutative property, we can say $x+y=y+x$ as much as $2+3=3+2$ or $4+3=3+4$. Similarly, $xy=yx$ is satisfied, as much as $3 \times 2 = 2 \times 3$. Similarly, as they satisfy the associative property, we can say $(x+y)+z=x+(y+z)$ as much as $(1+2)+3=1+(2+3)$. Similarly, they satisfy $a(b+c)=ab+ac$ as much as $2 \times (3+5) = 2 \times 3 + 2 \times 5$. Also, $(x+y)/z = x/z + y/z$ is satisfied.

Now, we can simplify expressions using these properties. For example,

$$2(3x+4)-4x = 2 \times 3x + 2 \times 4 - 4x \quad (1)$$

$$= (2 \times 3)x + 8 - 4x \quad (2)$$

$$= 6x + 8 - 4x \quad (3)$$

$$= 6x - 4x + 8 \quad (4)$$

$$= (6-4)x + 8 \quad (5)$$

$$= 2x + 8 \quad (6)$$

Of course, when you actually calculate such things, you don't need to write down the calculations step by step or line by line as in the above example. It takes too much time. Once you get used to this kind of calculations, perhaps you could simply get the answer as follows:

$$2(3x+4)-4x = 6x+8-4x \quad (7)$$

$$2(3x+4)-4x = 2x+8 \quad (8)$$

If you want to be a theoretical physicist, it is very important that you get used to such calculations. It is much more important than the ability to perform complicated arithmetic well, say, 23×18. Notice also what the above formula means: No matter what value you plug in for x, if you calculate $2(3x+4)-4x$, you are guaranteed to get $2x+8$ as answer. For example, if you plug in $x=1$, then $2 \times (3 \times 1 + 4) - 4 \times 1 = 10$, which is also equal to $2 \times 1 + 8$. So, to calculate $2(3x+4)-4x$ for $x=1$, we can just plug in $x=1$ to $2x+8$. The same goes for any other values for x. An equality such as (8) is called an "identity." No matter what value you plug in for x, the equality always holds.

We can sometimes use this property to check whether our calculation is wrong. This becomes useful when you take a test and have extra time after you have solved all the problems. For example, if you choose any value for x, say $x=0$, then $2(3x+4)-4x = 2 \times 4 - 4 \times 0 = 8$, while $2x+8 = 2 \times 0 + 8 = 8$. So, they are equal. So, our answer $2x+8$ seems to be correct. Of course, it is possible that we accidentally got the same value 8, even if our answer was wrong. On the other hand, if we got $2x+4$ for an answer, we can be sure that it is wrong as $2 \times 0 + 4$ is 4, which is different from 8.

A comment: $3x-2(x+6)$ is *not* equal to $3x-2x+12$, but instead is equal to $3x-2x-12$. This can be seen as follows:

$$3x-2(x+6) = 3x+(-2)(x+6) \quad (9)$$

$$= 3x+(-2)x+(-2)\times 6 \quad (10)$$

$$= 3x-2x+(-12) \quad (11)$$

$$= 3x-2x-12 \quad (12)$$

Similarly, $a-(b-c)$ is *not* $a-b-c$ but $a-b+c$. For example, $9-(7-3) = 9-4 = 5$ is *not* equal to $9-7-3 = -1$, but is equal to $9-7+3 = 2+3 = 5$.

Final comment. You might think that this method of expressing numbers by symbols is a mere formality. No, it isn't. As much as the discovery of zero as a placeholder allowed a tremendous innovation in arithmetic, so was the use of symbols for numbers. See the following English translation of passage from a book titled "Quesiti et Inventioni diverse" published in 1546 by an Italian mathematician Tartaglia.

> When the cube and things together
> Are equal to some number
> Find two others differing in this one
> Then you will keep as a rule
> That their product should always be equal

To the cube of a third of the things
The remainder then as a general rule
Of their cube roots subtracted
Will be your principal thing

In modern notation, this would be simply rendered as

If $x^3 + px = q$, find u, v such that

$$u - v = q, \qquad uv = \left(\frac{p}{3}\right)^3$$

Then, $x = \sqrt[3]{u} - \sqrt[3]{v}$.

It is much easier to understand this way.

Even though some Egyptians and Greeks used symbols for numbers, but in Europe, it was not until late in the 16th century that the French mathematician Viète suggested using symbols for numbers. Then, Descartes used symbols extensively in the 17th century influencing others.

Problem 1. Simplify the following expressions.

$$2(x+3) - 3(x-1) = ?, \qquad 4(y-3) - 5(x-y+2) = ? \tag{13}$$

Problem 2. Simplify the following expressions, assuming $z \neq 0$. (Hint[1])

$$\frac{x+2}{2} - \frac{x+3}{3} = ?, \qquad \frac{2z-y}{z} + \frac{y}{z} = ? \tag{14}$$

Problem 3. Choose a number, and add 3 to it. Then multiply the result by 2 and add 4. Then divide by 2 and subtract the number that you first chose. What number do you get? Explain why the answer is independent from the number you first chose. (Hint[2])

Summary

- We can use symbols to represent numbers. Such symbols obey all the properties of ordinary numbers.
- $7 \times x$ is usually represented as "$7x$" by omitting the multiplication sign. Similarly, ab means $a \times b$.
- $3x - 2(x+6)$ is equal to $3x - 2x - 12$. Similarly, $a - (b - c)$ is equal to $a - b + c$.

[1] Use $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ and $\frac{d}{f} + \frac{e}{f} = \frac{d+e}{f}$.

[2] Denote the number you chose as x and calculate $\{(x+3) \times 2 + 4\} \div 2 - x$. If the answer doesn't involve x, it should be independent from whatever number you first chose.