The calculation of π, the first part

In elementary school, you learned about π (pronounced as "pie"). It is the ratio of circumference to diameter of a circle, and is approximately given by 3.14. How do we know this value? According to the explanation in a Korean elementary school textbook, we can find the value by measuring the circumference of a circle and dividing it by the diameter. On the other hand, you may have heard that π is known up to trillions of digit. If mathematicians obtained the value of π by measuring the circumference, would they be able to obtain such an accurate value? Let's estimate how accurate π we can obtain by "measuring" it.

How big circle can we make? We live on the Earth, so it is very hard to make a circle bigger than the Earth. The circumference of the Earth is about 40000 km. So, it's safe to say that the maximum limit for the size of circle we can make is around 40000 km, even though it is much smaller in practice.

How accurately can we measure the circumference of a circle with 40000 km (4×10^7 meter)? It's hard to measure it more accurately than the size of an atom, which is about one hundred millionth centimeter (10^{-10} meter). Therefore, in this way, we can determine π only up to seventeen digits.

So, this certainly is not the way mathematicians find the value of π. But, I thought so when I read how the Ancient Greek mathematician Archimedes figured out in the 5th century B.C. that the value of π was

$$3\frac{10}{71}(\approx 3.1408) < \pi < 3\frac{1}{7}(\approx 3.1429). \tag{1}$$

The correct value is about 3.14159. Let me first explain Archimedes's method.

Archimedes considered the regular 96-gon inscribed in a circle to calculate the lower limit of π, and the regular 96-gon circumscribed about a circle to calculate the upper limit of π. See Fig. 1, if you do not understand the words "inscribe" and "circumscribe." A blue regular hexagon is inscribed in a circle, and a purple regular hexagon is circumscribed about a circle. Let's try to find π by using Archimedes's way, but this time, by using the regular hexagon, instead of the regular 96-gon.

Let's say that the radius of the circle is 1. See Fig. 2. Then the inscribed hexagon consists of six equilateral triangles with each side 1. As each side of

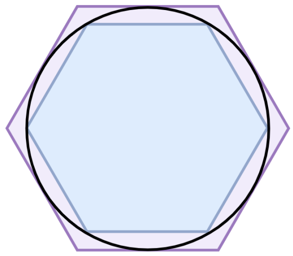

Figure 1: blue hexagon inscribed in a circle and purple hexagon circumscribed about a circle

the hexagon is 1, the perimeter of this hexagon is 6. See Fig. 3. In the case of the circumscribed hexagon, it consists of six equilateral triangles, each of which has the height 1. As the side of an equilateral triangle with height 1 is $2/\sqrt{3}$, the perimeter of the circumscribed hexagon is

$$\frac{2}{\sqrt{3}} \times 6 = 4\sqrt{3}. \tag{2}$$

The circumference of the circle is bigger than the perimeter of the inscribed hexagon (i.e., 6) and smaller than the perimeter of the circumscribed hexagon (i.e., $4\sqrt{3}$). Thus, the circumference to diameter ratio is given by

$$6 \div 2 = 3 < \pi < 2\sqrt{3}(= 4\sqrt{3} \div 2) \approx 3.46 \tag{3}$$

which is correct, considering that π is about 3.14.

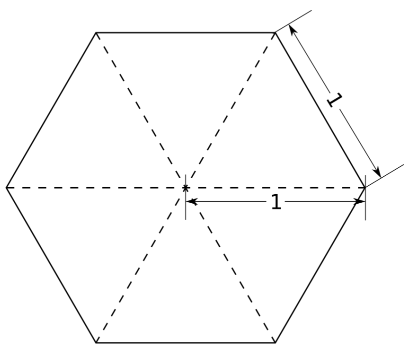

Figure 2: hexagon inscribed in a circle of radius 1

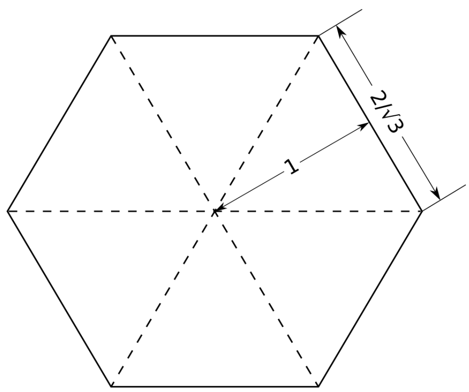

Figure 3: hexagon circumscribed about a circle of radius 1

What Archimedes did was finding the perimeter of the inscribed regular 96-gon and the perimeter of the circumscribed regular 96-gon. The circumference of the circle is between these two, as was the case with hexagons. So, why did he consider 96-gons instead of hexagons? Because the more sides a regular polygon has, the closer it is to a circle. Therefore, we can obtain a value that is closer to π.

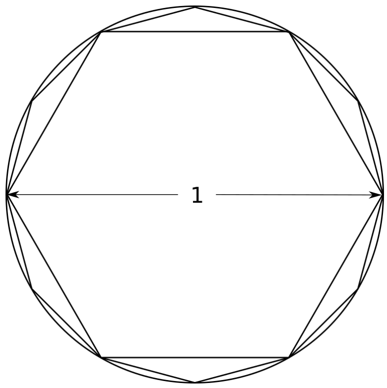

Figure 4: hexagon and dodecagon inscribed in a circle

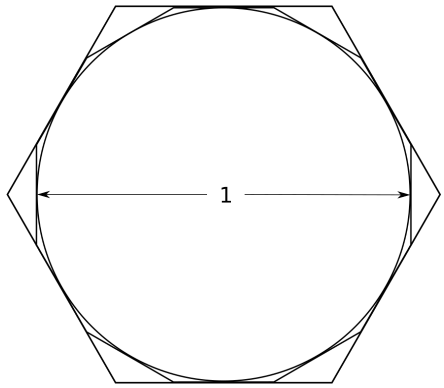

Figure 5: hexagon and dodecagon circumscribed about a circle

Actually, it is easy to see this. See Fig. 4 for the comparison of the perimeter of an inscribed regular dodecagon (a regular polygon with 12 sides) with the perimeter of an inscribed regular hexagon. The diameter of the circle is 1, which means that the circumference of the circle is π. From the figure, it is easy to see that

$$\text{perimeter of the hexagon} < \text{perimeter of the dodecagon} < \pi \tag{4}$$

See Fig. 5 for the comparison of the perimeter of an circumscribed regular dodecagon with the perimeter of a circumscribed regular hexagon. The diameter and the circumference are again 1 and π. Then, it is easy to see that

$$\pi < \text{perimeter of the dodecagon} < \text{perimeter of the hexagon} \tag{5}$$

Combining (4) and (5), we see that the dodecagons give a narrower range for π than the hexagons do. The similar conclusion can be drawn for the comparison between $2n$-gons and n-gons. Thus, $24(= 12 \times 2)$-gons give narrower range for π than dodecagons, $48(= 24 \times 2)$-gons give narrower range than 24-gons, and finally $96(= 48 \times 2)$-gons give narrower range than 48-gons.

When I first read that Archimedes had found the perimeters of these two regular 96-gons, I thought that he had measured them. I thought, measuring the sides of the regular 96-gons is perhaps easier than measuring the circumference of a circle. No, he hadn't. He obtained them by calculation, just as we obtained the perimeters of the two regular hexagons without measuring them. Archimedes found formulas to calculate the perimeters of inscribed and circumscribed regular $2n$-gons from the ones of n-gons. Understanding these formulas is not extremely hard. As any smart modern teenager who knows basic geometry, and understand the Pythagorean theorem can understand these formulas, we will explain them in the second part of this essay. Anyhow, Archimedes could thus obtain the perimeter of regular 12-gons, 24-gons, 48-gons, 96-gons.

In the 5th century, about 900 years after Archimedes, by considering the regular 24576-gon, the Chinese mathematician Zu Chongzhi correctly obtained a value of π that was between 3.1415926 and 3.1415927. If you double the number 96, eight times (i.e., or double the number 6, twelve times), you get 24576. In other words, the method Zu used was similar to Archimedes's method.

In the 15th century, the Indian mathematician Madhava made a breakthrough in calculating π, even though his method was not perhaps known in Europe before it was rediscovered there in the 17th century. Madhava derived the following relations:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \cdots \tag{6}$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \cdots \tag{7}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \tag{8}$$

where we are using radians for the angles.[1] If you are not familiar with the expression $\tan^{-1}$, please read our article "Inverse trigonometric functions."

He derived these relations without knowing the full-fledged calculus, even though he had discovered some ideas in calculus. Therefore, his derivation of these equations was quite complicated. In the 17th century, these equations were rediscovered in Europe, presumably without knowing his work. Now, they could be easily derived using the full-fledged calculus.

The important point is that Madhava used (8) to calculate π. Note that

$$\tan 45^\circ = \tan\left(\frac{\pi}{4}\right) = 1, \quad \Longrightarrow \quad \tan^{-1} 1 = \frac{\pi}{4} \tag{9}$$

Thus,

$$\pi = 4\tan^{-1} 1 = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots\right) \tag{10}$$

He obtained the above formula. However, he obtained another formula. Note that

$$\tan 30^\circ = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}, \quad \Longrightarrow \quad \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6} \tag{11}$$

[1] If you are not familiar with radians, please read our article "Radian."

Thus,

$$\pi = 6\tan^{-1}\frac{1}{\sqrt{3}} = \sqrt{12}\left(1 - \frac{1}{3\cdot 3^2} + \frac{1}{5\cdot 3^3} - \frac{1}{7\cdot 3^4} + \frac{1}{9\cdot 3^5} + \cdots\right) \tag{12}$$

By calculating the first 21 terms of the above expression, Madhava obtained a value of π that is correct to 11 decimal places.

Problem 1. What is the advantage of using (12) to calculate of π instead of (10)? Show that, if one uses the same number of terms, (12) yields a far more accurate value than (10). If you can, estimate roughly how many terms we need to add to obtain a value of π correct up to 10 digits, respectivley for (10) and for (12).

Later, many formulas that converge much more quickly have been discovered. There was no more need to consider the regular polygons. For example, in the 18th century, John Machin used the following formula

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} \tag{13}$$

to calculate π up to 100 digits.

In the early 20th century, Srinivasa Ramanujan discovered the following formula, which converges very quickly.

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}} \tag{14}$$

It computes a further eight decimal digits with each term. In 1988, Chudnovsky brothers discovered the following formula, which converges faster than Ramanujan's one. This formula was used to calculate π over trillion digits.

$$\frac{1}{\pi} = \frac{1}{426880\sqrt{10005}}\sum_{k=0}^{\infty}\frac{(6k)!(13591409 + 545140134k)}{(3k)!(k!)^3(-640320)^{3k}} \tag{15}$$

There are some formulas that are not quickly converging as these ones, but are beautiful. For example,

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \ddots}}} \tag{16}$$

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \ddots}}}} \tag{17}$$

In our earlier article, we mentioned that $\sqrt{2}$ is irrational. It goes without saying that π is irrational as well. Otherwise, the exact value would have been known, without ever needing to calculate π. In 1761, Johann Heinrich Lambert proved that π was irrational. Since then, several other proofs were discovered.

A comment. In 1621, Willebrord Snellius (known as "Snell" in English) obtained π correct up to seven digits, using 96-gons. Now, remember that Archimedes used the same 96-gons to obtain π correct only up to three digits (3.14). How could Snellius obtain such a more accurate value? I will explain how in the second part of this essay.