The calculation of π, the second part

When Archimedes calculated π, the mathematicians then didn't have the trigonometric functions as the ones which we use now, such as sine, cosine and tangent. However, as we now have them, let's approach Arichmedes's calculation of π using these trigonometric functions, because it will be easier to understand for modern readers, and we will get a fresh insight in analyzing Archimedes' calculation, as we will see.

Problem 1. Consider a regular n-gon inscribed in a circle with radius 1. Show that the length of each side is given by $2\sin(180°/n)$.

Problem 2. Consider a regular n-gon circumscribed about a circle with radius 1. Show that the length of each side is given by $2\tan(180°/n)$.

Thus, the perimeter of inscribed n-gon is $2n\sin(180°/n)$ and the perimeter of circumbscribed n-gon is $2n\tan(180°/n)$. Considering that the diameter is 2, we obtain that π, the circumference to the diameter ratio is given by

$$n\sin\frac{180°}{n} < \pi < n\tan\frac{180°}{n} \tag{1}$$

Thus, we see that what Archimedes essentially was plugging in $n = 96$ to the above equation. So, all he did was finding the sine and tangent values for the angle $180°/96$.

How did he find them? If I express what he essentially did in modern terminology, he used the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ to calculate $\sin(\theta/2)$ and $\tan(\theta/2)$. He started with hexagon, i.e., $\theta = 180°/6 = 30°$, as $\sin 30°$, $\cos 30°$, and $\tan 30°$ are well-known and quite simple. Then, he halved and halved the angles, until he reached $180°/96$. For the circumscribed polygon, he obtained $\tan(\theta/2)$ by using

$$\tan\frac{\theta}{2} = \frac{\tan\theta}{1/\cos\theta + 1} \tag{2}$$

For the inscribed polygon, he first obtained

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} \tag{3}$$

from which he deduced $\sin(\theta/2)$.

Now, notice that (3) is exactly the same thing (2). After obtaining $\tan(180°/96)$ for circumscribed 96-gon, he could have obtained $\sin(180°/96)$

directly from the value of $\tan(180°/96)$. But that's not what he did. He repeated the same calculation for the inscribed polygons. Why did he do so? He wanted to determine the sure range π falls. Thus, for the circumbscribed polygon, which gives the upper bound of π, he started out with the approximation of $\tan 30°$ that is slightly bigger than the actual value, and for the inscribed polygon, which gives the lower bound, he started out with the approximation of $\tan 30°$ that is slightly lower than the actual value. For example, in case of circumscribed hexagon, he used

$$\tan 30° < \frac{153}{265} \approx 0.577358 \tag{4}$$

and in case of inscribed hexagon, he used

$$\tan 30° > \frac{780}{1351} \approx 0.57735011 \tag{5}$$

The actual value is

$$\tan 30° = \frac{1}{\sqrt{3}} \approx 0.57735027 \tag{6}$$

Now, let me explain Archimedes' calculation, i.e., how he derived (2) and (3).

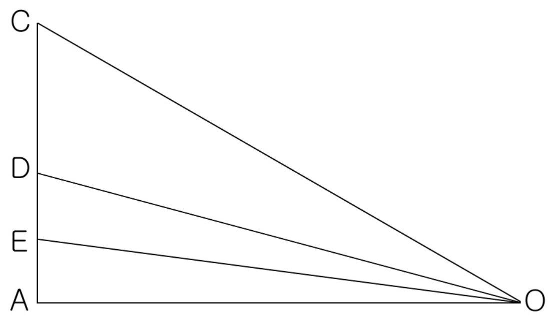

Figure 1: The goal is finding $\overline{DA}/\overline{OA}$ from $\overline{CA}/\overline{OA}$, when $\angle DOA = \angle COA/2$.

See Fig. 1. $\angle OAC$ is the right angle, $\angle AOC$ is 30°. Thus, we know

$$\overline{CA} : \overline{OA} : \overline{OC} = 1 : \sqrt{3} : 2 \tag{7}$$

In other words, we know the trigonometric functions for 30°. What we want to find is the trigonometric functions for 15°, i.e., the half of 30°. That is drawn by $\triangle DOA$. $\angle DOA$ is half of $\angle COA$. Thus, we see that $\angle COD$ is equal to $\angle DOA$. A well-known theorem in geometry is the following:

$$\overline{OC} : \overline{OA} = \overline{CD} : \overline{DA}, \qquad \text{if } \angle COD = \angle DOA \tag{8}$$

This is satisfied even though $\angle OAC$ is not the right angle. Anyhow, from the above relation, we can write

$$\frac{\overline{OC}}{\overline{OA}} = \frac{\overline{CD}}{\overline{DA}} \tag{9}$$

$$\frac{\overline{OC}}{\overline{OA}}+1 = \frac{\overline{CD}}{\overline{DA}}+1 \tag{10}$$

$$\frac{\overline{OC}+\overline{OA}}{\overline{OA}} = \frac{\overline{CD}+\overline{DA}}{\overline{DA}} \tag{11}$$

$$\frac{\overline{OC}+\overline{OA}}{\overline{OA}} = \frac{\overline{CA}}{\overline{DA}} \tag{12}$$

$$\frac{\overline{DA}}{\overline{OA}} = \frac{\overline{CA}}{\overline{OC}+\overline{OA}} \tag{13}$$

Thus, we obtained an expression for $\tan 15°$. We can simply plug the ratios in (7) into the right-hand side of (13). Archimedes repeated this process; for $\tan(15°/2)$, he simply replaced D in (13) by E, and C by D. Then, we have

$$\frac{\overline{EA}}{\overline{OA}} = \frac{\overline{DA}}{\overline{OD}+\overline{OA}} \tag{14}$$

We can plug the relevant ratios into the right-hand side of the above equation. However, we do not know yet $\overline{OD}$. Nevertheless, it can be easily deduced from Pythagorean theorem, i.e.,

$$\overline{OD} = \sqrt{\overline{OA}^2+\overline{DA}^2} \tag{15}$$

and so on.

Problem 3. Check that (13) indeed corresponds to (2).

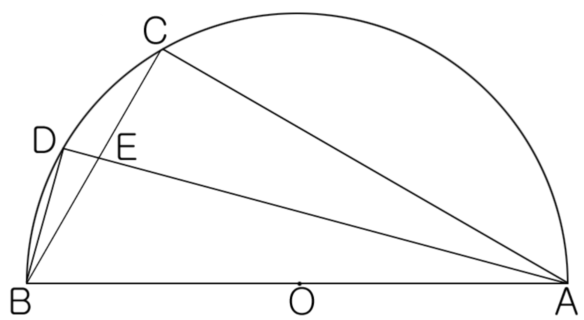

Figure 2: The goal is finding $\overline{DB}/\overline{AB}$ from $\overline{CB}/\overline{AB}$, when $\angle DAB = \angle CAB/2$.

Now, inscribed polygon. See Fig. 2. $\triangle ABC$ is a right triangle with $\angle CAB = 30°$. We know the ratio between the sides of this triangle very well. In our earlier language, if the radius of the circle is 1,

$$\overline{BC} = 2\sin\frac{180°}{6} = 1 \tag{16}$$

i.e., the side of inscribed hexagon. See Problem 1. What we want to find is $\overline{BD}$, which is 2sin $\frac{180°}{12}$, the side of inscribed dodecagon. We have

$$\angle DAB = \angle CAE = \angle DBE = 15° \tag{17}$$

Thus, we see three right triangles ($\triangle ADB$, $\triangle BDE$, $\triangle ACE$) with one of each angle 15°. They are all similar. Thus,

$$\frac{\overline{DB}}{\overline{AD}} = \frac{\overline{CE}}{\overline{AC}} \tag{18}$$

and

$$\overline{AB} : \overline{AD} = \overline{BE} : \overline{BD}, \qquad \rightarrow \qquad \frac{\overline{BE}}{\overline{AB}} = \frac{\overline{DB}}{\overline{AD}} \tag{19}$$

Combining (18) and (19), we obtain

$$\frac{\overline{DB}}{\overline{AD}} = \frac{\overline{CE}}{\overline{AC}} = \frac{\overline{BE}}{\overline{AB}} \tag{20}$$

$$\frac{\overline{DB}}{\overline{AD}} = \frac{\overline{CE}+\overline{BE}}{\overline{AC}+\overline{AB}} \tag{21}$$

$$\frac{\overline{DB}}{\overline{AD}} = \frac{\overline{BC}}{\overline{AC}+\overline{AB}} \tag{22}$$

Thus, we obtained a ratio of sides of the right triangle with one of its angle 15° in terms of the sides of the right triangle with one of its angle 30°. Or, more generally, the sides of the right triangle with one of its angle $\theta/2$ in terms of the sides of the right triangle with one of its angle θ. Notice that (22) is completely equivalent to (14). Anyhow, as was the case with circumscribed triangle, Archimedes obtains the ratio $\overline{AD} : \overline{DB} : \overline{AB}$ by Pythagorean theorem as follows:

$$\overline{AB} = \sqrt{\overline{AD}^2 + \overline{DB}^2} \tag{23}$$

Then, he repeatedly halved the angle by repeating (22), just as he did for circumscribed triangle.

Archimedes used 96-gon to obtain π correct up to three digits. However, as I mentioned in the first part of this essay, in 1621, Snellius obtained π correct up to seven digits, using 96-gon *as well.* How did he do it? He noticed that the perimeter of the inscribed polygon approaches on the circumference

n	$n\sin\left(\frac{180°}{n}\right)$	$n\tan\left(\frac{180°}{n}\right)$	$\pi - n\sin\left(\frac{180°}{n}\right)$	$n\tan\left(\frac{180°}{n}\right) - \pi$	ratio
6	3	3.46410	0.14159	0.32250	2.27772
12	3.10583	3.21539	0.03576	0.07380	2.06436
24	3.13263	3.15966	0.00896	0.01807	2.01553
48	3.13935	3.14609	0.00224	0.00449	2.00386

twice as fast as the perimeter of the circumscribed polygon. Let me explain what he meant.

Let's approach the problem of calculation of π using Archimedes's method. See the table above. In the first column, you see n. In the second column the lower bound of π, obtained using the inscribed polygon, and in the third column, the upper bound of π, obtained using the circumscribed polygon. If you remember the value of π, you will see that the lower bound is closer to π than the upper bound is. For example, for $n = 6$, 3 is closer to 3.14159, than 3.46410 is. Similarly, for $n = 12$, 3.10583 is closer to 3.14159 than 3.21539 is.

Could we systematically check this out? See the fourth column for the difference between π and its lower bound, and the fifth column for the difference between π and its upper bound. Indeed, the fourth column is quite smaller than the fifth column. How small is the fourth column compared to the fifth column? Or, how big is the fifth column compared to the fourth column? Let's divide the fifth column by the fourth column. We presented this ratio in the last column. You see that the ratio is around 2, and the bigger n the closer it is to 2. Actually, for $n = 48$, it is very close to 2. It is 2.00386. Will this trend continue for bigger n?

Snellius indeed believed that this trend will continue. He believed that this ratio would approach further closer to 2 for even bigger n, even though he couldn't explain why. So, he calculated $n\sin(180°/n)$ and $n\tan(180°/n)$ for $n = 96$, and figured out the value, which is twice closer to the former than the latter. That would be a good approximation for π. In mathematical formula, it can be obtained as follows:

$$\frac{2 \times 96\sin(180°/96) + 1 \times 96\tan(180°/96)}{2+1} = 3.14159283\cdots \qquad (24)$$

The real value for π is $3.14159265\cdots$. So, as I said, correct to seven digits.

In 1654, Christians Huygens indeed proved that the value of π obtained from the inscribed polygon indeed approaches π twice faster than the one obtained from the circumscribed polygon. Now, the calculation of π by Snellius had a concrete justification.

Huygens proved this a decade or two before calculus was invented. Therefore, even though I didn't look at his proof, I assume that it must be complicated, because he was not able to use easy tricks provided by calculus. Now, anyone who knows calculus can easily prove Snellius's observation. If you learn calculus, you will learn

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \tag{25}$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots \tag{26}$$

Problem 4. Show that (1) becomes the following formula, when the above expressions for sine and tangent functions are plugged in.

$$\pi - \frac{\pi^3}{6n^2} + \frac{\pi^5}{120n^4} - \cdots < \pi < \pi + \frac{\pi^3}{3n^2} + \frac{2\pi^5}{15n^4} + \cdots \tag{27}$$

Thus, the ratio we want to calculate is

$$\frac{n\tan\left(\frac{180^\circ}{n}\right) - \pi}{\pi - n\sin\left(\frac{180^\circ}{n}\right)} = \frac{\frac{\pi^3}{3n^2} + \frac{2\pi^5}{15n^4} + \cdots}{\frac{\pi^3}{6n^2} - \frac{\pi^5}{120n^4} + \cdots} = \frac{2 + \frac{4\pi^2}{5n^2} + \cdots}{1 - \frac{\pi^2}{20n^2} + \cdots} \tag{28}$$

Now, notice that $4\pi^2/(5n^2)$ in the numerator and $\pi^2/(20n^2)$ in the denominator are very small, when n is a big number. Furthermore, the terms denoted by $\cdots$ are even smaller. Thus, the terms other than 2 and 1 in the last expression are indeed negligible; the numerator approaches 2, and the denominator approaches 1. Therefore, we indeed see that the ratio approaches closer and closer to 2 for bigger and bigger n.

In our essay "The imagination in mathematics: "Pascal's triangle, combination, and the Taylor series for square root"," I mentioned Einstein's quote that raising new questions and possibilities requires creative imagination and marks real advance in science. In theoretical physics and math, there are still many observations like the one Snellius made 400 years ago. Currently, we do not know why they hold, but they still hold miraculously. They are waiting for proofs such as Huygens made three decades later.