The imagination in mathematics: "Pascal's triangle, combination, and the Taylor series for square root"

It is my impression that in popular belief, mathematicians and theoretical physicists are engaged in nothing more than very complicated and boring logical thinking. Indeed, I should confess that I myself held similar views when I was young. I thought that what theoretical physicists do was only matter of finding tricks to solve complicated differential equations. However, I have since discovered that such views are wrong, and I find it unfortunate that many people continue to hold them.

That these views are wrong has been clearly expressed by several renowned mathematicians and theoretical physicists. Edward Witten, the most renowned string theorist and winner of the Fields Medal (the mathematical equivalent of the Nobel Prize), said:

"Most people who haven't been trained in physics probably think of what physicists do as a question of incredibly complicated calculations, but that's not really the essence of it. The essence of it is that physics is about concepts, wanting to understand the concepts, the principles by which the world works."

In "the Evolution of Physics," Albert Einstein and Leopold Infeld wrote:

"The formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skill. To raise new questions, new possibilities, to regard old problems from a new angle, requires creative imagination and marks real advance in science." On another occasion, Albert Einstein said: "Logic will get you from A to B. Imagination will take you everywhere."

Einstein is telling us that theoretical physicists work by playing around with concepts and imagining them, rather than applying pure logic and calculation with no sense of what is actually going on behind the formulas. Of course, physicists do still need, eventually, to fill the gaps with logic and calculation but that doesn't diminish the importance of imagination in math and theoretical physics.

In this article, I will give you an example of how "imagination" and "regarding old problems from a new angle" can be useful in mathematics. Also in this example, I will show how mathematicians "fill the gaps with logic and calculation" after applying imagination.

1 Combination

To this end, I first introduce "Combination." Suppose that from a pool of 5 people designated A, B, C, D, and E you are appointing two people to equivalent officer positions. In how many possible ways can you choose the two officers?

Initially, you have five options for the first officer. Namely: A, B, C, D, and E.

Then, you will have four remaining options for the second officer. So you might conclude that there are 20=5×4 ways to choose the two officers. But if the positions are identical, there are redundancies. For instance, appointing C for the first officer and E for the second officer should be regarded the same as appointing E for the first officer and C for the second officer. There are two ways to permute any two officers. Therefore, we divide 20 by 2, and we get 10 unique officer selections.

Let's give you another example. Say that you are now choosing three co-presidents out of the five people A, B, C, D, and E. As in the previous example, suppose that three co-presidents are equivalent. Then how many possible ways can you choose the officers?

There are five options to choose from for the first co-president, then four options for the second co-president once one is already chosen, and finally three remaining options for the third co-president. So, we may conclude that there are 60=5×4×3 ways. But there are redundancies again.

For example, say all five people come to a meeting, and you're going to choose the first three who walk out the room to be co-presidents. There are 60 ways for the three persons to walk out of the room if we care about the order of these three co-presidents walking out of the room. However, even though these three may walk out of the room in different orders, they are still chosen to be the co-presidents. They can walk out of the room in any of six different orders.

Let's see where this number six comes from. Let's say that X, Y, Z are the three chosen co-presidents who first went out of the room. Then there are six possible orders for this set of three co-presidents to have walked out of the room. They are all equivalent in that the same three co-presidents are chosen.

First of all, three co-presidents could have been chosen to walk out of the room first. Namely: X, Y and Z.

Secondly, two co-presidents could have been chosen to walk out of the room second, since one co-president has already walked out of the room.

Thirdly, only one co-president could have been chosen to walk out of the room third, since two co-presidents have already walked out of the room. So, we conclude there are 6=3×2×1 equivalent possibilities. Namely, the following

$$(X, Y, Z)\ (X, Z, Y)\ (Y, X, Z)\ (Y, Z, X)\ (Z, X, Y)\ (Z, Y, X)$$

All these six are equivalent. So, if we divide $60 = 5 \times 4 \times 3$ by $6 = 3 \times 2 \times 1$, we get 10. In a mathematical language, we write this as $\binom{5}{3} = 10$ or ${}_5C_3 = 10$. In other words, the number of inequivalent choices of k unordered sets from n sets of object is written as $\binom{n}{k}$ or ${}_nC_k$. If you analyze the previous examples carefully, you will see that this is equal to $n!/k!/(n-k)!$. Here, $n!$ is pronounced "n factorial" and means "$1 \times 2 \times 3 \times \cdots (n-2) \times (n-1) \times n$."

Using this notation, we can write the first example as $\binom{5}{2} = 10$.

Surprisingly, the value for $\binom{5}{2}$ is equal to $\binom{5}{3}$. This is expected, since choosing two officers out of the five persons is the same thing as choosing three persons who are not going to be officers out of the five persons. Mathematically speaking, this can be written as

$$\binom{n}{k} = \binom{n}{n-k} \tag{1}$$

However, this was to be expected since $n!/k!/(n-k)!$ is equal to $n!/(n-k)!/[n-(n-k)]!$.

2 Pascal's triangle

Now, I introduce Pascal's triangle. Consider the expanding the following mathematical expressions, a process called binomial expansion:

We start with expressions $(a+b)^0$, $(a+b)^1$, $(a+b)^2$, $(a+b)^3$, $(a+b)^4$, and so on.

Expanding, we get[1]:

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{aligned}$$

and so on.

We notice that in general, one may write:

$$(a+b)^n = \sum_{k=0}^{n} f(n,k) a^k b^{n-k} \tag{2}$$

for some suitable functions $f(n,k)$, called "binomial coefficients," to be determined later. Now, let's write out the binomial coefficients in the following triangle, which is called Pascal's triangle:

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1
\end{array}$$

[1]Please read my article "Polynomials, expansion and factoring, if you are not familiar with this."

$$\begin{array}{c} 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \end{array}$$

and so on.

Equivalently:

$$\begin{array}{c} f(0,0) \\ f(1,0) \quad f(1,1) \\ f(2,0) \quad f(2,1) \quad f(2,2) \\ f(3,0) \quad f(3,1) \quad f(3,2) \quad f(3,3) \\ f(4,0) \quad f(4,1) \quad f(4,2) \quad f(4,3) \quad f(4,4) \\ f(5,0) \quad f(5,1) \quad f(5,2) \quad f(5,3) \quad f(5,4) \quad f(5,5) \end{array}$$

and so on.

Now, what can we see in this triangle? The left-most term and the right-most term of each row are always 1. This is to be expected, as we always have $f(n,0) = f(n,n) = 1$ from the definition of the functions $f(n,k)$. Moreover, if you are keen enough, you will see that the sum of any two adjacent numbers in the same row produces the number between these two numbers in the next row. In other words, the coefficients in the $(n+1)$th row of Pascal's triangle are sums of coefficients in the nth row. For example,

$$3 + 3 = 6$$

i.e. $f(3,1) + f(3,2) = f(4,2)$

$$4 + 1 = 5$$

i.e. $f(4,3) + f(4,4) = f(5,4)$

$$4 + 6 = 10$$

i.e. $f(4,1) + f(4,2) = f(5,2)$, etc.

The general formula that we obtain is:

$$f(n-1, k-1) + f(n-1, k) = f(n,k) \tag{3}$$

Why should this be so? Let's see how the sixth row can be obtained from the fifth row.

$$\begin{aligned}
&(a+b)^5 \\
&= (a+b)^4(a+b) \\
&= (a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)(b+a) \\
&= (a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + b^5) + (a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + ab^4) \\
&= a^5 + (1+4)a^4b + (4+6)a^3b^2 + (6+4)a^2b^3 + (4+1)ab^4 + b^5 \\
&= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
\end{aligned} \tag{4}$$

So the connection is clear. Given this, suppose you want to calculate $f(50,3)$. Might there be a simple way to obtain this? Or do you need to draw a large Pascal's triangle, adding and adding again many times over?

Here is where "imagination" enters the picture:

I claim:

$$f(n,k) = \binom{n}{k} \tag{5}$$

Here $\binom{n}{k}$ is what is commonly called "n choose k," the combination.

In other words, I claim that the coefficient of $a^k b^{n-k}$ in our example is precisely the number of the ways that we can choose k members out of n.

Let me show you why this is the case. As an example, we will consider the case $n=5$, $k=3$, i.e. $f(5,3)$. The generalization to arbitrary n and k will be straightforward.

Recall that $f(5,3)$ denotes the coefficient for a^3b^2 in

$$(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b)$$

Let's rewrite the above formula by introducing $a_1 = a_2 = a_3 = a_4 = a_5 = a$, $b_1 = b_2 = b_3 = b_4 = b_5 = b$.

We have immediately:

$$(a_1+b_1)(a_2+b_2)(a_3+b_3)(a_4+b_4)(a_5+b_5) = (a+b)^5$$

We notice immediately that we get a^3b^2 by choosing 3 "a"s out of 5 "a"s; the coefficient of a^3b^2 counts the number of ways to get there.[2] That is, in the expansion, we have the following terms equal to a^3b^2:

$$\begin{gathered}
a_1a_2a_3b_4b_5 \\
a_1a_2a_4b_3b_5 \\
a_1a_2a_5b_3b_4 \\
a_1a_3a_4b_1b_5 \\
a_1a_3a_5b_4b_5 \\
a_1a_4a_5b_2b_3 \\
a_2a_3a_4b_1b_5 \\
a_2a_3a_5b_1b_4 \\
a_2a_4a_5b_1b_3 \\
a_3a_4a_5b_1b_2
\end{gathered}$$

Again, these are the terms equal to a^3b^2. There are 10 such terms, so we have $f(5,3) = \binom{5}{3} = 10$. Notice that you would never have obtained this relation by pure logic alone,

[2]We do not need to separately count the number of ways to choose 2 "b"s out of 5"b"s, as bs are determined once as are chosen. Or, equivalently, you can forget about as and count the number of ways to choose 2 "b"s out of 5 "b"s. Either way, you get the same result as $\binom{n}{k} = \binom{n}{n-k}$.

without the aid of imagination. Pure logic tells you to draw a big Pascal's triangle by adding numbers over and over again. You won't get far by this method. You will only get from the nth row to the $(n+1)$th. These are the 'A' and 'B' of which Einstein spoke when he said, "Logic will get you from A to B." Using the original method, you would have had to compute all the values from the first row through the 49th before getting to the 50th row. Our new formula illustrates the second part of Einstein's quote, "Imagination will take you everywhere." We can now jump ahead to the 50th row using the concept of combination, skipping the other rows.

Now I would like to show you what is meant by "filling the gaps with logic and calculation." We still need to check the following equation in order to show rigorously that our new formula, $f(n,k) = \binom{n}{k}$ gives the same numbers as does the original method:

$$f(n-1,k-1) + f(n-1,k) = f(n,k) \tag{6}$$

We can prove this by using the explicit formula for $\binom{n}{k}$. This is easy.

$$\begin{aligned} &\binom{n-1}{k-1} + \binom{n-1}{k} \\ =& \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\ =& \frac{(n-1)!k}{k!(n-k)!} + \frac{(n-1)!(n-k)}{k!(n-k)!} \\ =& \frac{n!}{k!(n-k)!} \\ =& \binom{n}{k} \end{aligned} \tag{7}$$

In conclusion, we concretely showed

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \tag{8}$$

This is called "binomial theorem."

Finally, let's look at "raising new possibilities." Can we use our construction in the case that n is not equal to a non-negative integer? Remember, so far we have considered only the case that n is a non-negative integer such as 0, 1, 2, 3, and so on. The answer is yes. This was first considered by Isaac Newton around 1665. Let's consider $n = 1/2$, $a = 1$, $b = x$. Then by our formula, we have the following:

$$\sqrt{1+x} = (1+x)^{1/2} = 1 + \binom{1/2}{1} x + \binom{1/2}{2} x^2 + \binom{1/2}{3} x^3 \cdots \tag{9}$$

and so on (it will become clear why this continues indefinitely).[3] You will be invited to prove the above formula, when you learn Taylor series.

[3]Of course, in this case, we cannot put the coefficients in the form of triangle, because n is not a positive integer. Still, we can use the formula.

Now, there seems to be a difficulty in that the formula for combination includes factorials, but we do not know how to calculate $(1/2)!$.

Don't worry. Notice that for non-negative integers, we have seen that the following alternative formulas are equivalent to the ones introduced in Section 1:

$$\begin{aligned}\binom{n}{1} &= n \\ \binom{n}{2} &= \frac{n(n-1)}{2\times 1} \\ \binom{n}{3} &= \frac{n(n-1)(n-2)}{3\times 2\times 1}\end{aligned} \tag{10}$$

and so on...

These new formulas make sense even when n is not an integer.

Therefore we have, for example:

$$\begin{aligned}\binom{1/2}{1} &= 1/2 \\ \binom{1/2}{2} &= -1/8 \\ \binom{1/2}{3} &= 1/16\end{aligned}$$

And so we have:

$$\sqrt{1+x} = 1 + x/2 - x^2/8 + x^3/16 + \cdots \tag{11}$$

Notice that the above expression never terminates, as no matter how large k in $\binom{1/2}{k}$ may be, the coefficient never equals zero. This is in contrast to the earlier case in which n is non-negative integer. In that case, the higher coefficients would simply all have been zero. The expansion for $(1+x)^n$ terminates with x^n; there is no factor of x^{n+1}. For example, even though we write $(1+x)^2$ as a non-terminating series as follows,

$$(1+x)^2 = \binom{2}{0} + \binom{2}{1}x + \binom{2}{2}x^2 + \binom{2}{3}x^3 + \binom{2}{4}x^4 + \cdots \tag{12}$$

$\binom{2}{3}$ is zero, as one can check from (10). As all the coefficients for higher order terms are also zero, the series terminates at x^2 order.

Back to the present example, it is important to notice that the terms with high powers of x are very small, when x is small enough. Moreover, the higher the power of x, the faster the term gets small. This allows us, practically speaking, to ignore such terms when x is sufficiently close to 0. For example, if $x = 0.01$ we have:

$$\sqrt{1+0.01} = 1 + 0.01/2 - 0.0001/8 + 0.000001/16 + . \tag{13}$$

If you evaluate $\sqrt{1.01}$ using a calculator you will get $1.0049875621121\cdots$ while $1+0.01/2-0.0001/8+0.000001/16 = 1.0049875625$. You see that our formula, even if we stop after a couple terms, is close enough! If we were to consider the x^4 term, the difference would be even smaller. It turns out that we can always get as close as we want to the exact value just by adding up more terms. This kind of approach, in which we evaluate a certain number by adding up more and more terms that are getting smaller and smaller, is very useful in physics and very widely used. Actually, this is how calculators calculate sine, cosine, and tangent functions. If you are curious about this and want to learn more about this approach, study calculus and learn Taylor series in our article "Taylor series."

Problem 1. Expand $(a-c)^5$ by using (4). (Hint[4])

Problem 2. Show the following for $x \ll 1$.

$$(1+x)^n \approx 1 + nx \tag{14}$$

You should remember this formula.

Problem 3. Show the following for $v \ll c$. This problem will play an important role when we later talk about Einstein's theory of relativity.

$$\frac{1}{\sqrt{1-v^2/c^2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots \tag{15}$$

(Hint[5])

Problem 4. Show the following for $E^2 \gg m^2$.(Hint[6])

$$\sqrt{E^2 - m^2} \approx E - \frac{m^2}{2E} + \cdots \tag{16}$$

This relation will be used in our article "Neutrino oscillation, clarified."

Problem 5. Prove the following identities:

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n, \qquad \sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0 \tag{17}$$

(Hint.[7]) For example, when $n = 4$ we have

$$1+4+6+4+1 = 16, \qquad 1-4+6-4+1 = 0 \tag{18}$$

3 More general cases

This section will be only useful for what is called "Maxwell-Boltzmann distribution" later in our article "The Bose-Einstein distribution, the Fermi-Dirac distribution and the Maxwell-Boltzmann distribution." In the first reading, you can skip it.

Suppose that you divide a pool of 11 people into three groups called L, M and N with 5 people, 4 people and 2 people respectively. In how many possible ways can you divide?

[4] Use $(a-c)^5 = (a+(-c))^5$. This implies that we can just plug in $-c$ for b in (4).

[5] Use $\frac{1}{\sqrt{1-v^2/c^2}} = (1-v^2/c^2)^{-1/2}$

[6] Use $\sqrt{E^2-m^2} = \sqrt{E^2(1-m^2/E^2)} = m(1-m^2/E^2)^{1/2}$.

[7] Use $(1+1)^n = 2^n$ and $((-1)+1)^n = 0$, respectively for (8).

Let's say that you choose the member of L first. There is $\binom{11}{5}$ ways to do so. Then, you have $11 - 5$ people left. You can now choose the members of M. There are $\binom{11-5}{4}$ ways to do so once the members of L are chosen. Now you have $11 - 5 - 4 = 2$ people left. They are now the members of N. So, in total, we have the following number of ways to choose the three groups:

$$\binom{11}{5}\binom{11-5}{4} = \frac{11!}{5!(11-5)!}\frac{(11-5)!}{4!2!} = \frac{11!}{5!4!2!} \tag{19}$$

This is the answer. Of course, the answer doesn't depend on in which order you choose the members of group. For example, if you first choose N (2 people) then, L (5 people), then N (4 people), we have:

$$\binom{11}{2}\binom{11-2}{5} = \frac{11!}{2!5!4!} \tag{20}$$

which is exactly same as (19). This agreement should not be surprising, considering that we have already encountered a similar one in (1).

We just considered the case in which we divide people into three groups, but this can be easily generalized to arbitrary numbers of groups. For example, if you divide 50 people to 5 groups called P, Q, R, S and T with 8 people, 8 people, 12 people, 10 people, 10 people respectively, the number of possible ways is given by

$$\frac{50!}{8!8!12!10!10!} \tag{21}$$

We can also see that the "combination" we considered in section 1, is just a special case of formulas such as (19) and (21); there, we just had two groups instead of three or five.

Problem 6. What is the coefficient of $a^2b^2c^2$ in the following expansion?

$$(a+b+c)^6 \tag{22}$$

Problem 7. From the definition of factorial, we must have $n! = n \times (n-1)!$. From this relation, deduce what $0!$ must be, and show that your deduction correctly leads to $\binom{n}{n}$=1. (i.e., there is only one way to choose n people from n people.)

A comment. Factorial may not seem to be defined for non-integers or negative numbers. For example, we know $1! = 1$, $2! = 2\times1 = 2$, $3! = 3\times2\times1 = 6$ and so on but what would $\frac{1}{2}!$ be? How about $(-\frac{1}{2})!$? The definition for the factorial you know doesn't provide any immediate ways to calculate these values. But, in our later article "Gamma function," you will learn that there are ways to define the factorial in other ways which are also applicable to non-integers or negative numbers. In that article, you will be invited to prove that $(-\frac{1}{2})! = \sqrt{\pi}$.

Problem 8. Given that $(-\frac{1}{2})! = \sqrt{\pi}$, calculate the following:

$$\left(\frac{1}{2}\right)! =?, \qquad \left(-\frac{3}{2}\right)! =? \tag{23}$$

(Hint[8])

[8]Use $n! = n \times (n-1)!$.