The imagination in mathematics: “Pascal’s triangle, combination, and the Taylor series for square root”

It is my impression that in popular belief, mathematicians and theoretical physicists are engaged in nothing more than very complicated and boring logical thinking. Indeed, I should confess that I myself held similar views when I was young. I thought that what theoretical physicists do was only matter of finding tricks to solve complicated differential equations. However, I have since discovered that such views are wrong, and I find it unfortunate that many people continue to hold them.

That these views are wrong has been clearly expressed by several renowned mathematicians and theoretical physicists. Edward Witten, the most renowned string theorist and winner of the Fields Medal (the mathematical equivalent of the Nobel Prize), said:

“Most people who haven’t been trained in physics probably think of what physicists do as a question of incredibly complicated calculations, but that’s not really the essence of it. The essence of it is that physics is about concepts, wanting to understand the concepts, the principles by which the world works.”

In “the Evolution of Physics,” Albert Einstein and Leopold Infeld wrote:

“The formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skill. To raise new questions, new possibilities, to regard old problems from a new angle, requires creative imagination and marks real advance in science.” On another occasion, Albert Einstein said: “Logic will get you from A to B. Imagination will take you everywhere.”

Einstein is telling us that theoretical physicists work by playing around with concepts and imagining them, rather than applying pure logic and calculation with no sense of what is actually going on behind the formulas. Of course, physicists do still need, eventually, to fill the gaps with logic and calculation but that doesn’t diminish the importance of imagination in math and theoretical physics.

In this article, I will give you an example of how “imagination” and “regarding old problems from a new angle” can be useful in mathematics. Also in this example, I will show how mathematicians “fill the gaps with logic and calculation” after applying imagination.

To this end, I will introduce Pascal’s triangle. Consider the expanding the following mathematical expressions, a process called binomial expansion:

We start with expressions $(a+b)^0$, $(a+b)^1$, $(a+b)^2$, $(a+b)^3$, $(a+b)^4$, and so on.

Expanding, we get[1]:

$$
\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{aligned}
$$

and so on.

We notice that in general, one may write:

$$(a+b)^n = \sum_{k=0}^{n} f(n,k)a^k b^{n-k} \tag{1}$$

for some suitable functions $f(n,k)$, called "binomial coefficients," to be determined later. Now, let's write out the binomial coefficients in the following triangle, which is called Pascal's triangle:

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1
\end{array}
$$

and so on.

Equivalently:

$$
\begin{array}{c}
f(0,0) \\
f(1,0) \quad f(1,1) \\
f(2,0) \quad f(2,1) \quad f(2,2) \\
f(3,0) \quad f(3,1) \quad f(3,2) \quad f(3,3) \\
f(4,0) \quad f(4,1) \quad f(4,2) \quad f(4,3) \quad f(4,4) \\
f(5,0) \quad f(5,1) \quad f(5,2) \quad f(5,3) \quad f(5,4) \quad f(5,5)
\end{array}
$$

and so on.

Now, what can we see in this triangle? The left-most term and the right-most term of each row are always 1. This is to be expected, as we always have $f(n,0) = f(n,n) = 1$ from

[1] Please read my article "Polynomials, expansion and factoring."

the definition of the functions $f(n,k)$. Moreover, if you are keen enough, you will see that the sum of any two adjacent numbers in the same row produces the number between these two numbers in the next row. In other words, the coefficients in the $(n+1)$th row of Pascal's triangle are sums of coefficients in the nth row. For example,

$$3+3=6$$

i.e. $f(3,1)+f(3,2)=f(4,2)$

$$4+1=5$$

i.e. $f(4,3)+f(4,4)=f(5,4)$

$$4+6=10$$

i.e. $f(4,1)+f(4,2)=f(5,2)$, etc.

The general formula that we obtain is:

$$f(n-1,k-1)+f(n-1,k)=f(n,k) \tag{2}$$

Why should this be so? Let's see how the sixth row can be obtained from the fifth row.

$$\begin{aligned}
&(a+b)^5\\
&=(a+b)^4(a+b)\\
&=(a^4+4a^3b+6a^2b^2+4ab^3+b^4)(b+a)\\
&=(a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5)+(a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4)\\
&=a^5+(1+4)a^4b+(4+6)a^3b^2+(6+4)a^2b^3+(4+1)ab^4+b^5\\
&=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{aligned}$$

So the connection is clear. Given this, suppose you want to calculate $f(50,3)$. Might there be a simple way to obtain this? Or do you need to draw a large Pascal's triangle, adding and adding again many times over?

Here is where "imagination" enters the picture:

I claim:

$$f(n,k)=\binom{n}{k} \tag{3}$$

Here $\binom{n}{k}$ is what is commonly called "n choose k," the combination. If you do not know what $\binom{n}{k}$ is, please read my article, "Combination."

In other words, I claim that the coefficient of a^kb^{n-k} in our example is precisely the number of the ways that we can choose k members out of n.

Let me show you why this is the case. As an example, we will consider the case $n=5$, $k=3$. The generalization to arbitrary n and k will be straightforward.

Recall that $f(5,3)$ denotes the coefficient for a^3b^2 in

$$(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b)$$

Let's rewrite the above formula by introducing $a_1 = a_2 = a_3 = a_4 = a_5 = a$, $b_1 = b_2 = b_3 = b_4 = b_5 = b$.

We have immediately:

$$(a_1+b_1)(a_2+b_2)(a_3+b_3)(a_4+b_4)(a_5+b_5) = (a+b)^5$$

We notice immediately that we get a^3b^2 by choosing 3 "a"s out of 5 "a"s; the coefficient of a^3b^2 counts the number of ways to get there. That is, in the expansion, we have the following terms equal to a^3b^2:

$$a_1a_2a_3b_4b_5$$
$$a_1a_2a_4b_3b_5$$
$$a_1a_2a_5b_3b_4$$
$$a_1a_3a_4b_1b_5$$
$$a_1a_3a_5b_4b_5$$
$$a_1a_4a_5b_2b_3$$
$$a_2a_3a_4b_1b_5$$
$$a_2a_3a_5b_1b_4$$
$$a_2a_4a_5b_1b_3$$
$$a_3a_4a_5b_1b_2$$

Again, these are the terms equal to a^3b^2. There are 10 such terms, so we have $f(5,3) = \binom{5}{3} = 10$. Notice that you would never have obtained this relation by pure logic alone, without the aid of imagination. Pure logic tells you to draw a big Pascal's triangle by adding numbers over and over again. You won't get far by this method. You will only get from the nth row to the $(n+1)$th. These are the 'A' and 'B' of which Einstein spoke when he said, "Logic will get you from A to B." Using the original method, you would have had to compute all the values from the first row through the 49th before getting to the 50th row. Our new formula illustrates the second part of Einstein's quote, "Imagination will take you everywhere." We can now jump ahead to the 50th row using the concept of combination, skipping the other rows.

Now I would like to show you what is meant by "filling the gaps with logic and calculation." We still need to check the following equation in order to show rigorously that our new formula for $f(n,k)$ gives the same numbers as does the original method:

$$f(n-1,k-1) + f(n-1,k) = f(n,k) \tag{4}$$

We can prove this by using the explicit formula for $\binom{n}{k}$. This is easy.

$$\begin{aligned}
&\binom{n-1}{k-1} + \binom{n-1}{k} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\
&= \frac{(n-1)!k}{k!(n-k)!} + \frac{(n-1)!(n-k)}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k} \qquad (5)
\end{aligned}$$

Finally, let's look at "raising new possibilities." Can we use our construction in the case that n is not equal to a non-negative integer (remember, so far we have considered only the case that n is a non-negative integer such as 0,1,2,3, and so on)? The answer is yes. Let's consider $n = 1/2$, $a = 1$, $b = x$. Then by our formula, we have the following:

$$\sqrt{1+x} = (1+x)^{1/2} = 1 + \binom{1/2}{1}x + \binom{1/2}{2}x^2 + \binom{1/2}{3}x^3 \cdots \qquad (6)$$

and so on (it will become clear why this continues indefinitely).

Now, there seems to be a difficulty in that the formula for combination includes factorials, but we do not know how to calculate (1/2)!.

Don't worry. Notice that for non-negative integers, the following alternative formulas are equivalent to the old ones:

$$\begin{aligned}
\binom{n}{1} &= n \\
\binom{n}{2} &= \frac{n(n-1)}{2 \times 1} \\
\binom{n}{3} &= \frac{n(n-1)(n-2)}{3 \times 2 \times 1}
\end{aligned}$$

and so on...

These new formulas make sense even when n is not an integer.

Therefore we have, for example:

$$\begin{aligned}
\binom{1/2}{1} &= 1/2 \\
\binom{1/2}{2} &= -1/8 \\
\binom{1/2}{3} &= 1/16
\end{aligned}$$

And so we have:

$$\sqrt{1+x} = 1 + x/2 - x^2/8 + x^3/16 + \cdots \qquad (7)$$

Notice that the above expression never terminates, as no matter how large k in $\binom{1/2}{k}$ may be, the coefficient never equals zero. This is in contrast to the earlier case in which n is non-negative integer. In that case, the higher coefficients would simply all have been zero. The expansion for $(1+x)^n$ terminates with x^n; there is no factor of x^{n+1}. This is why we got away with ignoring them before.

Back to the present example, it is important to notice that the terms with high powers of x are very small, when x is small enough. Moreover, the higher the power of x, the faster the term gets small. This allows us, practically speaking, to ignore such terms when x is sufficiently close to 0. For example, if $x = 0.01$ we have:

$$\sqrt{1+0.01} = 1 + 0.01/2 - 0.0001/8 + 0.000001/16 + . \tag{8}$$

If you evaluate $\sqrt{1.01}$ using a calculator you will get $1.0049875621121\cdots$ while $1+0.01/2-0.0001/8+0.000001/16 = 1.0049875625$. You see that our formula, even if we stop after a couple terms, is close enough! If we were to consider the x^4 term, the difference would be even smaller. It turns out that we can always get as close as we want to the exact value just by adding up more terms. This kind of approach, in which we evaluate a certain number by adding up more and more terms that are getting smaller and smaller, is very useful in physics and very widely used. Actually, this is how calculators calculate sine, cosine, and tangent functions. If you are curious about this and want to learn more about this approach, study calculus and learn Taylor series.

Problem 1. Show the following for $v \ll c$. This problem will play an important role when we later talk about Einstein' theory of relativity.

$$\frac{1}{\sqrt{1-v^2/c^2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots \tag{9}$$

(Hint[2])

Problem 2. Prove the following identities:

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n, \qquad \sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0 \tag{10}$$

(Hint.[3]) For example, when $n = 4$ we have

$$1+4+6+4+1 = 16, \qquad 1-4+6-4+1 = 0 \tag{11}$$

[2] Use $\frac{1}{\sqrt{1-v^2/c^2}} = (1-v^2/c^2)^{-1/2}$

[3] Use $(1+1)^n = 2^n$ and $(1-1)^n = 0$ respectively.