Linear inequalities

In the last article, we have explained how to solve linear equations. We have seen that we can solve linear equations by using the property that the equality still holds even when we add, subtract, multiply, and divide the both sides by the same number.

In this article, we will explain how to solve linear inequalities. The strategy is similar, but we have to be more careful; while the inequality doesn't change when you add or subtract by the same number, or multiply or divide by the same number that is positive, the inequality *changes* the direction when you multiply or divide by a negative number. Let me explain all these with examples.

First, addition by the same number. If we have $5 > 3$, we have $5 + 2 > 3 + 2$. We can add a negative number, too. $5 + (-2) > 3 + (-2)$. Similarly, if we have $3 < 5$, we have $3 + 2 < 5 + 2$ and $3 + (-2) < 5 + (-2)$.

Second, subtraction by the same number. If we have $5 > 3$, we have $5 - 4 > 3 - 4$. We can also subtract a negative number. $5 - (-1) > 3 - (-1)$. Similarly, if we have $3 < 5$, we have $3 - 4 < 5 - 4$ and $3 - (-1) < 5 - (-1)$.

Third, multiplication by a positive number. If we have $5 > 3$, we have $5 \times 2 > 3 \times 2$. Similarly, if we have $3 < 5$, we have $3 < 5 \times 2$.

Fourth, division by a positive number. If we have $5 > 3$, we have $5 \div 2 > 3 \div 2$. Similarly, if we have $3 < 5$, we have $3 \div 2 < 5 \div 2$.

Now comes the important point. Let's see what happens if we multiply a negative number, -2 on $5 > 3$. We see that $5 \times (-2) > 3 \times (-2)$ is *not* satisfied. We have $5 \times (-2) < 3 \times (-2)$. So, we see that multiplication by a negative number changes the inequality. Similarly, if we have $3 < 5$, $3 \times (-2) > 5 \times (-2)$.

The inequality also changes when you divide by a negative number, because dividing by a negative number is equivalent multiplying by (another) negative number. For example, dividing by -2 is equivalent to multiplying by $-1/2$.

Let's look at again why the inequality sign changes. Let's say $x > y$. Then, we have

$$x - x - y > y - x - y \tag{1}$$

$$-y > -x \tag{2}$$

$$-x < -y \tag{3}$$

where in the last step, we used $a > b$ implies $b < a$. So, we started with $x > y$, and ended up with $-x < -y$. We indeed see that multiplying by -1 changes the inequality. Notice that this is satisfied regardless of the sign of x and y. Let's check it case by case. For example, if

x and y are positive, we have $5 > 3$ implies $-5 < -3$. If x is positive and y is negative, we have $2 > -4$ implies $-2 < 4$. If x and y are both negative, we have $-4 < -3$ implies $4 > 3$.

Now, some actual examples. Let's solve the following.

$$\begin{aligned} x+3 &< 2x-4 && (4)\\ x-2x &< -4-3 && (5)\\ -x &< -7 && (6)\\ x &> 7 && (7) \end{aligned}$$

Of course, we can solve this slightly differently.

$$\begin{aligned} x+3 &< 2x-4 && (8)\\ 3+4 &< 2x-x && (9)\\ 7 &< x && (10)\\ x &> 7 && (11) \end{aligned}$$

Either way, you get the same answer.

Another example. Let's solve $x+4 < -x+2 < 5$. We need to solve $x+4 < -x+2$ and $-x+2 < 5$. We get $x < -1$ and $x > -3$, respectively. So, the answer is $-3 < x < -1$.

Another example. Let's solve $x < 2x < -x-4$. We need to solve $x < 2x$ and $2x < -x-3$. We get $x > 0$ and $x < -1$. Since there is no x that satisfies both $x > 0$ and $x < -1$, there is no solution to these inequalities.

Final comment. So far, we only talked about $>$ and $<$, but the observations made in this article regarding $>$ and $<$ can be equally applied to $\geq$ and $\leq$. For example, if we have $x \leq y$, we necessarily have $-x \geq -y$.

Problem 1. Solve the following inequality.

$$-x-3 > x-5$$

Problem 2. Solve the following inequality.

$$x-4 \geq 3x-12$$

Problem 3. Solve the following inequality.

$$5x-4 < 4x+3$$

Problem 4. Solve the following inequalities.

$$x+3 < 2x \leq 3x$$

Problem 5. Solve the following inequalities.

$$x+4 < 3x < 2x$$

Problem 6. Solve the following inequalities.

$$4 \leq x < -x + 6$$

Problem 7. If $a < b$ and $1/a > 1/b$, what can we say about the signs of a and b? (Hint[1])

Problem 8. This problem is about Seiberg duality. Of course, you need to know very advanced physics (supersymmetry) to understand Seiberg duality, but you can still check a couple of the evidences for Seiberg duality with very easy math. Seiberg duality states that a theory with the number of flavor N_f with the number of color N_c and its dual theory (which we denote with $'$) with the number of flavor $N'_f = N_f$ and the number of color $N'_c = N_f - N_c$ are the same theory in low energy. (If you are curious about what flavor and color mean, you can read "Pauli's exclusion principle, Color Charge of Quarks, Asymptotic freedom and Confinement" but it won't help you to solve this problem at all.) When the following is satisfied

$$\frac{3}{2}N_c < N_f < 3N_c \tag{12}$$

the original theory has infrared fixed point. Check that this is the same condition as the following condition

$$\frac{3}{2}N'_c < N'_f < 3N'_c \tag{13}$$

in which the dual theory of the original theory has infrared fixed point. Also, check $N_f = N'_f$ and $N_c = N'_f - N'_c$. This means that the dual theory of the dual theory is the original theory. Of course, the demonstration of Seiberg duality is not so easy as presented in this problem, as these are not the only evidences of Seiberg duality.

Summary

- A linear inequality can be solved by using a strategy that you use when you solve linear equations.
- However, you have to be careful. While the inequality doesn't change when you add or subtract by the same number, or multiply or divide by the same number that is positive, the inequality *changes* the direction when you multiply or divide by a negative number.
- To solve inequalities of form $a < b < c$, you need to solve $a < b$ and $b < c$, and find the overlapping range of the two solutions.

[1] Divide $a < b$ by ab.