Logarithm

Logarithm is a very important concept in mathematics and physics. Logarithm (also called "log") is defined as follows:

$$\text{if } a^b = c, \qquad \text{we have : } \log_a c = b \tag{1}$$

where we call a "a base." Now, let us give you some examples:

$$\log_{10} 1000 = 3 \quad (\text{ as } 10^3 = 1000), \qquad \log_4 2 = \frac{1}{2} \quad (\text{ as } 4^{\frac{1}{2}} = 2) \tag{2}$$

Now some properties of log. First, notice the following:

$$a^b \times a^c = a^{b+c} \tag{3}$$

If we say $a^b = f$, $a^c = g$, we have $a^{b+c} = fg$, which implies:

$$b = \log_a f, \qquad c = \log_a g, \qquad b + c = \log_a (fg) \tag{4}$$

Therefore we conclude:

$$\log_a f + \log_a g = \log_a (fg) \tag{5}$$

Let's derive another identity for logarithm using this formula. If we let $fg = h$, $g = h/f$, we have:

$$\log_a f + \log_a \frac{h}{f} = \log_a h \tag{6}$$

which in turn implies:

$$\log_a h - \log_a f = \log_a \frac{h}{f} \tag{7}$$

Let's derive still another identity. Notice following

$$(a^m)^n = a^{mn} \tag{8}$$

If we let $a^m = p$ and we have:

$$\log_a p = m, \qquad \log_a (p^n) = mn \tag{9}$$

which implies:

$$\log_a (p^n) = n \log_a p \tag{10}$$

In engineering, logarithm with base 10 called "common logarithm" is useful. Many even go on to omit the marker for the base in such a case. For example, $\log 10000 = 4$, $\log 0.01 = -2$.

In math and physics, it turns out that logarithm with choice of base a number called "e" is useful. In mathematics, e is as important number as π and given by $2.718\cdots$. We call logarithm with base e "natural log" and denote it by ln after French "logarithm naturel." For example, $\ln e = \log_e e = 1$. Some mathematicians and physicists use the symbol log in place of ln. For example $\log e = 1$. Therefore it could be sometimes confusing whether log means $\log_{10}$ or ln.

In any case, why the number e is special and often used as the base for logarithm is explained in my article "Exponential function and natural log."

Problem 1. (Hint[1])

$$\log_8 4 = ?, \qquad \log_3 8 + \log_3 \frac{9}{8}, = ? \qquad \log_{1/2} 4 = ?, \qquad e^{2\ln 3} = ?, \qquad e^{3\ln 2} = ? \tag{11}$$

Problem 2. Prove the following. (Hint[2])

$$\log_a b = \log_a c \cdot \log_c b \tag{12}$$

Notice that this implies $\ln b = \ln 10 \cdot \log_{10} b \approx 2.303 \log_{10} b$

Problem 3. Prove the following.(Hint[3]) Notice that this is useful when calculating the values for $\log_a b$ where a is neither 10 or e, since most calculators provide log values only for base 10 and e.

$$\log_a b = \frac{\log_{10} b}{\log_{10} a} = \frac{\ln b}{\ln a} \tag{13}$$

Problem 4. Let's say that a population of a certain country grows by 7 percents every year. Then, find out how long it takes for it to double, using the logarithm button in a scientific calculator such as the one provided by Microsoft Windows. If you do it correctly you should get about 10 years. Repeat the calculation for the annual growth 5 percents, 3 percents, 2 percents and 1 percent. If you do it correctly, you should get about 14 years, 23 years, 35 years, 70 years respectively. Notice that the annual growth rate multiplied by the years that takes the population to double is roughly given by 70. For example, $7 \times 10 \approx 5 \times 14 \approx 3 \times 23 \approx 2 \times 35 \approx 1 \times 70 \approx 70$. This is known as "law of seventy," which I learned from an economics textbook. Using this law, one can easily calculate how long a quantity with fixed annual growth rate takes to double. For example, if an annual growth rate is 4 percents, it would take about $18(\approx 70 \div 4)$ years to double. However, the law of seventy does not hold for big annual growth rate. For example, if the annual growth rate is 100 percent, it takes exactly a year to double, not $0.7(= 70 \div 100)$ year to double. Anyhow, in our later article "Exponential function and natural log," you will be invited to prove law of seventy. You will need to use the natural log.

[1] For the first one, try to use (8) with n the answer we want, $a = 2$, $a^m = 8$ and $a^{mn} = 4$.

[2] Let $a^x = b$, $a^y = c$ and $c^z = b$, and try to derive $x = yz$

[3] This is the solution to $a^x = b$. Obtain the solution by taking $\log_{10}$ or ln on both-hand sides. You will need to use (10).