Manifold

Manifold is an important concept in mathematics. However, you don't usually get a chance to learn about manifolds, unless you take courses such as "Analysis on Manifolds" which is usually taken by 4th year math major undergraduates. Nevertheless, the basic concept is not difficult, and more importantly, one can learn about the power of mathematics, that mathematics is about imagination, extending what we can see into what we can't.

Let's begin with 1-dimensional manifold. A line is 1-dimensional, as each point on a line can be located by a single number. For example, let's say you take a point in the line as the origin. Then, you can assign the number 2.5 to another point if it is on a certain side of the origin and 2.5 cm away from the origin. If another point is on the opposite side of the origin and 3 cm away from the origin, you can assign the number -3. Therefore, we say a line is a 1-dimensional manifold.

A plane is 2-dimensional as a point on a plane can be located by two numbers. Think of 2-dimensional Cartesian coordinate system. Therefore, it is a 2-dimensional manifold.

Similarly, a 3-dimensional object which has a volume is called a 3-dimensional manifold as three numbers are needed to pinpoint a point in it.

The idea is similar for higher-dimensions, even though we cannot visualize them. We need n numbers to locate a point in n-dimensional manifold.

Now comes the question. What are the other examples of 1-dimensional manifold? A good example of 1-d manifold is a circle. A circle is defined by the set of points which are on a 2-dimensional plane and equidistant to the center of the circle. In an earlier article, we have seen that a circle can be described by the formula $(x - x_0)^2 + (y - y_0)^2 = r^2$. See also Fig.1. A point on a circle such as B can be located by a single number, the angle "θ." For example, the θ for A is zero. Notice that θ is periodic. If θ is 360°, 720° or 1080°, it all describes the same point A, and therefore, the period of θ is 360°.

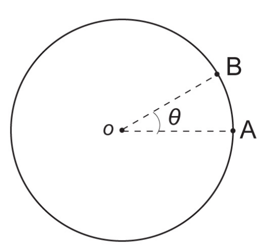

Figure 1: a circle

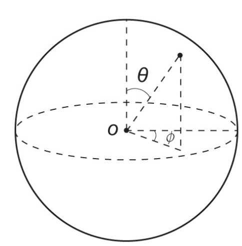

Figure 2: a sphere

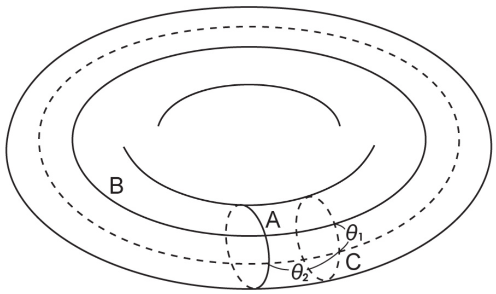

Figure 3: a torus

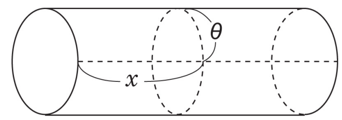

Figure 4: a cylinder

How about the other examples of 2-dimensional manifold? A good one is a sphere. A sphere is defined by the set of points which are on a 3-dimensional space and equidistant to the center of the sphere. It can be described by $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$. Also, we all know that we need two numbers to locate a point: latitude θ and longitude ϕ. See Fig.2. You are familiar with this if you know how to locate a point on Earth which looks similar to a sphere. Here, I want to emphasize that a "sphere" in mathematical terminology is somewhat different from a sphere in our daily life. To make an analogy, if you have a orange, the peel is the sphere and not the interior which you eat, which is called "ball" or, more precisely speaking, "open ball." (We will come back to it, soon)

Now, following the logic regarding the circle, and the sphere, we can think about n-dimensional sphere, which is called "n-sphere" and denoted S^n. For example, a circle is S^1 and a usual sphere (i.e. the two-dimensional one) is S^2. Following this logic, one can easily see that S^3 is a 3-dimensional sphere described by the equation $(w - w_0)^2 + (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$ in 4-dimensional Cartesian coordinate system (w, x, y, z). Of course, it is hard to draw a 3-sphere on a sketchbook, nor is it easy to imagine what it looks like. Nevertheless, we can concretely deal with 3-sphere mathematically.

Another good example of 2-dimensional manifold is a torus. Again, we need two numbers to locate a point on torus. See Fig.3. We have θ_1 and θ_2 as coordinates to locate the point C. The solid lines here can be regarded as axes. We say a torus is a direct product of two circles, as to pinpoint a point in a torus, we need two periodic angles, θ_1 and θ_2. In other words, if you have two circles A and B, and use θ_1 to locate a point in the first circle and use θ_2 to locate a point in the second circle, there is one to one correspondence between a location on a torus and the set of each location of a point on A and of a point on B. Therefore, torus in two-dimension, denoted as T^2, is given by $T^2 = S^1 \times S^1$.

Notice also that S^2 is *not* $S^1 \times S^1$, as when θ the latitude is 0 (i.e. "the North Pole,") no matter what ϕ the longitude is, they lead to the same North pole point. On the other hand, in case of torus, even though $\theta_1 = 0$ if θ_2 is all different, they lead to different points. So, there is no one to one correspondence between $S^1 \times S^1$ and S^2.

Another good example of 2-dimensional manifold is the 2-ball, the interior of a circle. Here, 2 in 2-ball denotes the fact that 2-ball is a 2-dimensional manifold. 2-ball is sometimes called a "disk." Open disk is collections of points (x, y) satisfying

$$(x - x_0)^2 + (y - y_0)^2 < r^2 \tag{1}$$

Closed disk is collections of points (x, y) satisfying

$$(x - x_0)^2 + (y - y_0)^2 \leq r^2 \tag{2}$$

Similarly, the interior of a sphere is called "3-ball" or, more precisely speaking, "open 3-ball." To make an our earlier analogy with an orange, open 3-ball is the interior part you eat. Open 3-ball is collections of points (x, y, z) satisfying

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < r^2 \tag{3}$$

Closed 3-ball is collections of points (x, y, z) satisfying

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \leq r^2 \tag{4}$$

This corresponds to the whole orange. Similarly, the interior of n-sphere is called "open $(n + 1)$-ball," and if you include the surface it is called "closed $(n + 1)$-ball."

Another good example of 2-dimensional manifold is a cylinder. We need two numbers to locate a point in a cylinder. One number θ to point out the location along the circle, and another number x to point out the location along the flat direction. Therefore, a cylinder is given by $S^1 \times \mathbb{R}^1$ where $\mathbb{R}^1$ is a line. See Fig.4.

Then, it is easy to see that a plain plane is given by $\mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1$. Think of it as x-y plane in Cartesian coordinate system. In general, $\mathbb{R}^n$ for a positive integer n, is called "Euclidean space."

Before concluding this article, I would like to make three comments. First, in our example, n-sphere was represented as a part of $n+1$ dimensional Euclidean space; we used $n+1$ numbers in Cartesian coordinate system to describe n-sphere. This is an example of "embedding." We say n-sphere is embedded into $\mathbb{R}^{n+1}$. Similarly, in our description, n-ball was embedded into $\mathbb{R}^n$. Notice that the case with sphere and the case with ball is slightly different. In case of sphere, the embedding was done to a higher-dimensional manifold. However, there are ways to describe manifolds such as spheres without embedding to higher-dimensional manifold, but on its own. As we can locate a point in 2-sphere with two numbers (θ, ϕ) instead of three numbers (x, y, z), there are ways to locate a point in 3-sphere with three numbers instead of four numbers (w, x, y, z). Moreover, we can still know whether the manifold is curved without thinking about this embedding. Let me put it this way. For example, let's say there is an imaginative animal that cannot sense 3-dimensions, but only 2-dimensions. Would it be able to know the fact that the shape of the Earth is roughly a sphere? It can, as it will come back to the starting point, if it keeps traveling in any arbitrary direction, as long as it doesn't change direction. Actually, the distance it has to travel to come back to the starting point doesn't depend on the direction. Moreover, as we will see in our later article "Non-Euclidean geometry," if you draw a circle on a sphere, the ratio of circumference to the diameter will be less than π, and the sum of the angles of any triangle will be bigger than $180°$. Similarly, if our Universe is a 3-sphere, which is a serious possibility considered by cosmologists, if a spaceship keeps traveling in any arbitrary, yet fixed, direction, it will come back to Earth after a certain distance that doesn't depend on the direction. Same can be said about the ratio of circumference to the diameter and the sum of the angles of any triangle; the ratio of circumference to the diameter will be less than π, and the sum of the angles of any triangle will be bigger than $180°$. Of course, as mentioned, 3-sphere is hard to visualize for human beings.

Second comment. Classifying manifolds in various dimensions is important in mathematics. It is known that 4-manifolds are the most difficult to classify, as 4 is a number that is neither small nor big. For low dimensions, the classification is easy because low-dimensional manifolds are simpler. For high-enough-dimensional manifolds, mathematicians can use some strategies that work generally for such high dimensions. But, such strategies don't work for 4-manifold, because the number 4 is not that big. Nor is 4 small enough to classify 4-manifolds as we can do with lower-dimensional manifolds. Anyhow, 4-dimensional manifolds play an important role in string theory. This is one of the areas which string theory contributed most to mathematics.

Third comment. We know that our space we live in has three dimensions. You need three numbers to specify a point in the space. However, if we add the time to this 3-dimensional space, our spacetime is 4-dimensional. To make an appointment to meet your friend, you need four numbers. Three numbers to denote where to meet (i.e. latitude, longitude on the Earth, and the height, like the floor of a building) and one number to denote when. However, string theorists believe that the spacetime in our universe is 11-dimensional. One dimension for time and 10 dimensions for space. We think that our 11-dimensional universe manifold is something like $\mathbb{R}^4 \times M_7$ where $\mathbb{R}^4$ is our usual 4-dimensional spacetime we live in and M_7 is the extra seven dimensional manifold, or simply "extra dimensions." (If our Universe is 3-sphere, it should be $S^3 \times \mathbb{R}^1 \times M_7$ where S^3 is the space we live in and $\mathbb{R}^1$ denotes time.) But, don't we see only 4-dimensional spacetime? Where are the extra dimensions M_7? String theorists believe that they are so small that we cannot feel it. To make an analogy, if S^1 in the cylinder $\mathbb{R}^1 \times S^1$ is very small, the cylinder will approximately look like a line $\mathbb{R}^1$. String theorist and popular science writer Brian Greene put it this way. If you look at this extremely thin garden hose, it looks something like a 1-dimensional line. However, if you look at it closely with magnifying lens, you then see the circular part of the hose. Similarly, the extra-dimensional part M_7 in $\mathbb{R}^4 \times M_7$ can be seen only when we look at it closely. In our human eyes or everyday experience, we can only see $\mathbb{R}^4$ part of $\mathbb{R}^4 \times M_7$.

Through hard work of string theorists, some mathematical properties of the extra dimensions were deduced, but at this point it is far from clear how exactly the extra dimensions must look like, even though they found “T-dualities” among five different string theories, as we have mentioned in “M-theory and dualities.” Some string theorists work the other way around. They investigate how the extra dimensions must look like in order to match our physics laws, in particular the forces in our universe, such as gravity force, electromagnetic force, strong force and weak force. This approach is known as “bottom-up approach.”

String theorists were not the first ones who came up with the idea of extra dimensions. In the early 20th century, Kaluza and Klein showed that Einstein’s general relativity in 5-dimensional spacetime (precisely speaking $\mathbb{R}^4 \times S^1$, i.e. the extra dimension is a circle) reproduces Einstein’s general relativity in 4-dimensional spacetime and electromagnetism. Their idea got some attention when their work was first published, but it got more attention in the late 20th century as it was then found that string theory necessarily requires extra dimensions. We will teach you Kaluza and Klein’s ideas in our later article “Kaluza-Klein theory,” but you have a long way to go before learning all its prerequisites!

Problem 1. Would a bug on an infinitely long cylinder come back to its starting point if it keeps going in any arbitrary, yet fixed, direction like the one on sphere does? If not, which direction must it be headed to come back to the starting point? Denote the direction in Fig.4.

Problem 2. If our Universe is indeed 3-sphere, it means that our Universe is finitely large, because you will come back to your original point after traveling a finite distance. In this case, can there be an edge of our Universe? By “edge” I mean the 2-dimensional surface of our Universe which divides the space into what is inside our Universe and what is outside our Universe. (Hint[1])

Summary

- A line is a 1-dimensional manifold, and a plane is a 2-dimensional manifold.
- We need n numbers to locate a point in n-dimensional manifold.
- n-dimensional sphere is called “n-sphere” and denoted as S^n. For example, a circle is S^1.

[1] Think whether a 2-sphere has an edge. An edge of 2-dimensional object, if existing, should be 1-dimensional.