The mathematical definition of vector

Mathematically speaking, a vector is an entity that satisfies the following eight conditions.

1) Vector addition must be associative. (i.e. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$)

2) Vector addition must be commutative. (i.e. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$)

3) Vector addition must have an identity element. (i.e. For any vector $\vec{u}$, there exists an identity element $\vec{e}$ that satisfies $\vec{u} + \vec{e} = \vec{u}$. Namely $\vec{e} = 0$)

4) Vector addition must have inverse elements. (i.e. $\vec{u} + (-\vec{u}) = 0$, where $-\vec{u}$ is the additive inverse of $\vec{u}$.

5) Distributivity must hold for scalar multiplication over vector addition. (i.e. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$)

6) Distributivity must hold for scalar multiplication over field addition. (i.e. $(a + b)\vec{u} = a\vec{u} + b\vec{u}$)

7) Scalar multiplication must be compatible with multiplication in the field of scalars. (i.e. $a(b\vec{u}) = (ab)\vec{u}$)

8) Scalar multiplication must have an identity element. (i.e. $1\vec{u} = \vec{u}$)

It is easy to check that our earlier notion of vector satisfies all the above conditions. For example, if we denote three three-dimensional vectors as follows:

$$\vec{u} = u_x\hat{x} + u_y\hat{y} + u_z\hat{z}, \quad \vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}, \quad \vec{w} = w_x\hat{x} + w_y\hat{y} + w_z\hat{z} \tag{1}$$

Then, the first condition is satisfied since:

$$\begin{aligned}
\vec{u} + (\vec{v} + \vec{w}) &= u_x\hat{x} + u_y\hat{y} + u_z\hat{z} + ((v_x + w_x)\hat{x} + (v_y + w_y)\hat{y} + (v_z + w_z)\hat{z}) \\
&= (u_x + v_x + w_x)\hat{x} + (u_y + v_y + w_y)\hat{y} + (u_z + v_z + w_z)\hat{z} \\
(\vec{u} + \vec{v}) + \vec{w} &= ((u_x + v_x)\hat{x} + (u_y + v_y)\hat{y} + (u_z + v_z)\hat{z}) + w_x\hat{x} + w_y\hat{y} + w_z\hat{z} \\
&= (u_x + v_x + w_x)\hat{x} + (u_y + v_y + w_y)\hat{y} + (u_z + v_z + w_z)\hat{z}
\end{aligned}$$

The fourth condition is also satisfied since:

$$-\vec{u} = -u_x\hat{x} - u_y\hat{y} - u_z\hat{z} \tag{2}$$

returns zero when added to $\vec{u}$. One can check the other conditions also easily.

Final remark. In this article, we saw that an array of three numbers can be regarded as a three-dimensional vector. For example, an array of three real number can be regarded as three-dimensional real vector, and a set of such vectors is called three-dimensional real vector space, and denoted as $\mathbb{R}^3$. Similarly, an array of n complex numbers can be regarded as an n-dimensional complex vector, which is an element of $\mathbb{C}^n$, the n-dimensional complex vector space. We will later see that infinite-dimensional complex vector space is useful for quantum mechanics.

Summary

- Mathematically, vectors are quantities that satisfie certain reasonable linearity conditions.
- An array of n real numbers can be regarded as a n-dimensional real vector.
- An array of n complex numbers can be regarded as a n-dimensional complex vector.