Neutrino oscillation, clarified

In an earlier article "Neutrino oscillation," I explained neutrino oscillation at a layman's level. However, I couldn't explain it fully as the knowledge of quantum mechanics was not assumed. In this article, I will explain neutrino oscillation using quantum mechanics.

Neutrino oscillation occurs because the flavor eigenstates of neutrinos are not equal to their mass eigenstates; they are related by unitary matrices as follows:

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} \tag{1}$$

where ν_e, ν_μ and ν_τ denote electron neutrino, muon neutrino, and tau neutrino respectively, and ν_1, ν_2 and ν_3 denote mass eigenstates and U denotes the unitary matrix that relates them. This matrix is called the Pontecorvo-Maki-Nakagawa-Sakata matrix (PMNS matrix), Maki-Nakagawa-Sakata matrix (MNS matrix), lepton mixing matrix, or neutrino mixing matrix.

(**Problem 1.** Show that the PMNS matrix has to be a unitary matrix if both the flavor eigenstates and the mass eigenstates are properly normalized.)

The above expression can be re-expressed as:

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle \tag{2}$$

where $|\nu_\alpha\rangle$ denotes the flavor eigenstates, $|\nu_i\rangle$ the mass eigenstates, and $U_{\alpha i}$ the PMNS matrix.

Now, we will consider the case in which neutrinos are flying in free space (i.e. with the potential being zero) and see how the wave functions evolve. From "A short introduction to quantum mechanics XI: comparison with de Broglie's matter waves and the time-dependent Schrödinger equation," the wave function evolves as follows:

$$|\nu_i(t)\rangle = e^{-i(E_i t - \vec{p}_i \cdot \vec{x})} |\nu_i(t=0)\rangle \tag{3}$$

where we have used the natural units. Also, we know that $E_i^2 = m^2 + p_i^2$ where we have used the natural units again.

Given this, remember that the masses of neutrinos are very small. This implies:

$$p_i = \sqrt{E_i^2 - m^2} \approx E_i - \frac{m^2}{2E_i} \tag{4}$$

Also, it implies that the neutrinos travel at a speed almost that of light. Therefore, the distance traveled by neutrinos during the time interval t is t in natural units. Plugging this value and (4) to (3), we get:

$$|\nu_i(t)\rangle = e^{-im_i^2 t/2E_i} |\nu_i(t=0)\rangle = e^{-im_i^2 t/2E} |\nu_i(t=0)\rangle \tag{5}$$

where in the last step we have assumed that the energy of neutrinos from the same source is independent of its type and for convenience called the energy "E."

From the above formula, it is obvious that the number of neutrinos of a given mass eigenstate doesn't change as $\langle\nu_i(t)|\nu_i(t)\rangle = \langle\nu_i(0)|\nu_i(0)\rangle$. However, the same cannot be said for the number of neutrinos of a certain flavor. To see this, first observe:

$$|\nu_\alpha(t)\rangle = \sum_i U_{\alpha i} |\nu_i(t)\rangle = \sum_i \sum_\beta U_{\alpha i} U^*_{\beta i} e^{-im_i^2 t/2E} |\nu_\beta(0)\rangle \tag{6}$$

(**Problem 2.** Derive the above formula. Hint[1])
As there is no guarantee that the transition amplitude from $|\nu_\alpha(0)\rangle$ to $|\nu_\alpha(t)\rangle$ (i.e. $\langle\nu_\alpha(0)|\nu_\alpha(t)\rangle = \sum_i U_{\alpha i} U^*_{\alpha i} e^{-im_i^2 t/2E}$) is a pure phase, there is similarly no guarantee that the initial ν_α would remain unchanged to other flavors. Therefore, we see that the number of ν_α is not conserved for a generic PMNS matrix. It is also easy to see that the probability that neutrino changes its flavor (i.e. ν_β changes to ν_α for $\alpha \neq \beta$) is non-zero for a generic PMNS matrix, as there is no guarantee that $\langle\nu_\beta(0)|\nu_\alpha(t)\rangle = \sum_i U_{\alpha i} U^*_{\beta i} e^{-im_i^2 t/2E}$ is zero. This is the crux of neutrino oscillation. The flavor of neutrino changes.

Problem 3. Show that there would be no neutrino oscillation if all three mass eigenvalues of neutrinos were same. (Hint[2]) Another way of seeing this is that in such a case the flavor eigenstate would be the mass eigenstate since eigenstates of such a mass matrix (i.e. a matrix that is a scalar multiple of an identity matrix) can be any state. Remember that the number of neutrinos of a given mass eigenstate doesn't change.

The discussion so far seems somewhat abstract without explicit further calculations. However, for three neutrino oscillation, such explicit calculations can be too complicated to briefly explain here. As a hypothetical case in which there exist only two neutrino flavors, we can catch all the qualitatively features of neutrino oscillation. We will now consider explicit calculations for such a case.

[1] Use the unitarity of the PMNS matrix to show that (2) implies $|\nu_i(0)\rangle = \sum_\beta U^*_{\beta i} |\nu_\beta(0)\rangle$

[2] Use $\sum_i U_{\alpha i} U^*_{\beta i} = \delta_{\alpha\beta}$.

Let the 2-dimensional analog of (1) be given as follows:

$$\begin{bmatrix} \nu_\alpha \\ \nu_\beta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \tag{7}$$

Here, θ is called "mixing angle." Apparently the PMNS matrix is unitary. Then, one can show (**Problem 5.**) the probability amplitude by which ν_β will transform into ν_α is given by

$$\begin{aligned} P(\beta \to \alpha) &= |\langle \nu_\beta(0)|\nu_\alpha(t)\rangle|^2 = \langle \nu_\beta(0)|\nu_\alpha(t)\rangle \langle \nu_\beta(0)|\nu_\alpha(t)\rangle^* \\ &= (\cos\theta\sin\theta)^2 \left(2 - 2\cos\frac{(m_1^2 - m_2^2)t}{2E}\right) \\ &= \sin^2(2\theta)\sin^2\left(\frac{(m_1^2 - m_2^2)t}{4E}\right) \end{aligned} \tag{8}$$

Therefore, we see clearly why neutrino flavor changing is called neutrino oscillation. Rather than changing only in one direction, the flavor of neutrinos change back and forth as time elapses as is clear from the term $\sin^2(\frac{(m_1^2-m_2^2)t}{4E})$. Also, notice that we can only know the difference of the mass squared from the neutrino oscillation; we can find neither the mass itself nor even which ones are bigger.

Let me conclude this article by providing you with some explicit values obtained from experiments and observations to give you some sense. For three flavors, there are three mixing angles: θ_{12}, θ_{13}, θ_{23}. Here, θ_{12} is the mixing angle between ν_1 and ν_2 and so on. The mixing angles are measured to be,

$$\begin{aligned} \sin^2(2\theta_{12}) &= 0.85 \pm 0.02 \\ \sin^2(2\theta_{23}) &= 0.99 \pm 0.01 \\ \sin^2(2\theta_{13}) &= 0.09 \pm 0.01 \end{aligned}$$

The differences of squared masses are measured to be,

$$\begin{aligned} |m_1^2 - m_2^2| &= (7.5 \pm 0.2) \times 10^{-5}\text{eV}^2 \\ |m_2^2 - m_3^2| &= (2.5 \pm 0.1) \times 10^{-3}\text{eV}^2 \end{aligned}$$

Summary

- Neutrino flavor oscillates (i.e. the number of neutrinos with a certain flavor changes) as the neutrino flavor eigenstate is different from the neutrino mass eigenstate. This phenomenon is due to quantum mechanical time evolution of phase. Of course, the number of neutrino with a definite mass doesn't change.

- The neutrino oscillation gives us only the information about the differences of squared masses of neutrinos, not the absolute value of the masses.