Non-Euclidean geometry

In our earlier article "Manifold," we introduced the concept of manifold. There, we stated that $\mathbb{R}^n$ was the Euclidean space. Precisely speaking, this was not a correct statement. Let me give you the correct statement. If you take two vectors in the Euclidean space $\mathbb{R}^n$, say $\vec{x} = (x_1, x_2, \cdots, x_n)$ and $\vec{y} = (y_1, y_2, \cdots, y_n)$ then the inner product is given by

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \tag{1}$$

Notice that if $\vec{z}$ is a non-zero vector, then

$$\vec{z} \cdot \vec{z} > 0 \tag{2}$$

The Euclidean space always admits the coordinate system which satisfy (1). Such a coordinate system is called the Cartesian coordinate system which you are familiar with. I say "admit" because there are other coordinate systems, which do not satisfy (1), but you can use to describe the same Euclidean space. Polar coordinate and spherical coordinate are good examples. Of course, even though (1) is re-defined in the ways appropriate for the polar coordinate and the spherical coordinate, (2) is always satisfied.

Now, let's talk about the Minkowski space. The Minkowski space (sometimes called "the Lorentzian space") is also given by $\mathbb{R}^n$, but the dot product of two vectors, say $\vec{x} = (x_1, x_2, \cdots, x_n)$ and $\vec{y} = (y_1, y_2, \cdots, y_n)$ is given by

$$\vec{x} \cdot \vec{y} = -x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \tag{3}$$

If you read our earlier article "Rotation and the Lorentz transformation, orthogonal and unitary matrices," it should be clear why dot product must be defined this way in Minkowski space. Notice also that (2) is not always satisfied. Again, if a space admits the coordinate system which satisfy (3), it's the Minkowski space.

The Euclidean space and the Minkowski space are examples of manifolds with no curvature. They are not curved at all. A good example of curved manifold would be spheres. Let's first look at S^2 (i.e. 2-sphere, or simply sphere) closely, and what the rules of the geometry look like on 2-sphere. This is an example of what is called "non-Euclidean geometry." Certainly 2-sphere is not Euclidean space.

Figure 1: a triangle

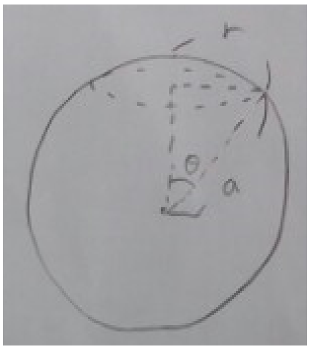

Figure 2: a circle

What would a straight line in a sphere look like? To find the answer, think of the sphere as the Earth, as the Earth looks approximately like a sphere. Let's say you choose a point on a sphere. Without loss of generality, let's find a coordinate system that defines the North Pole as the point you chose. Then, any direction from the North Pole is southward. If you keep moving, without changing the direction, you will always reach the South Pole. Once you reach the South Pole, if you keep going in the same direction, the direction will be northward, and you will come back to the North Pole. More generally, if you start at a certain point, and if you keep going straight, you will reach the "antipodal point" of the original point, then come back to the original point. The antipodal point of a point is the point directly opposite to it. For example, the antipodal point of the North Pole is the South Pole.

In the Euclidean space, we know that there is only one straight line that is parallel to another certain straight line and passes a certain point. (Parallel lines are defined by two lines that never meet.) However, there is no such a parallel straight line on sphere. Let's see why it is so. Suppose you first have a straight line. You know that this straight line divides a sphere to two parts. For example, the equator, which is a straight line, divides the Earth into the Northern hemisphere and the Southern hemisphere. Then, let's pick a point A. If the point A is on the Northern hemisphere, the antipodal point of A must be on the Southern hemisphere. Any straight line that passes A must pass these two points. Since the antipodal point is in the Southern hemisphere, the straight line has to pass the equator twice: when it crosses from the Northern hemisphere to the Southern hemisphere and when it crosses from the Southern hemisphere to the Norther hemisphere. Same statement can be made when A is on the Southern hemisphere and its antipodal point is on the Northern hemisphere. So, in a sphere, any two straight lines alway meet. Therefore, there can be no parallel lines. Another interesting property of straight lines in a sphere is that the sum of the angles inside a triangle is always bigger than $180°$. See Fig. 1 for an

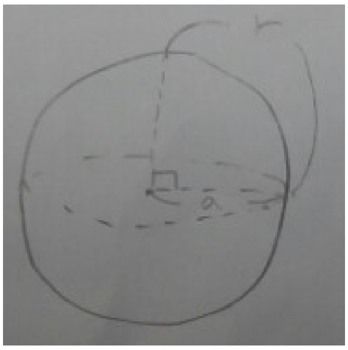

Figure 3: the biggest circle

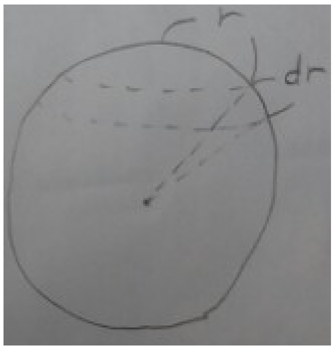

Figure 4: the area of a circle

example. One of the vertices of the triangle is located at the North Pole, and the other two are located at the equator, and the length of the three sides is all same. In this case, the sum is 270°. It is also easy to imagine that if the triangle is smaller, the effect of curvature is smaller, making the sum of the angles in triangle closer to 180°. Think it along this way. If you confine your movement in your city, and draw a triangle, you will hardly notice that the Earth is round and the sum of the angles in your triangle will be most likely close to 180°. Actually, it is mathematically proven that the bigger the area of the triangle the bigger the sum of the angles of the triangle. Also, it is actually shown that if two triangles have the same area the sum of the angles inside them are same regardless of their shape.

Also, I want to mention that on spheres the ratio of the circumference of a circle to the radius is smaller than 2π. See Fig. 2. Let's see quantitatively. Let's say the radius of sphere is a and the radius of a circle inside the sphere is r. Then, the angle θ in the figure is given by $\theta = r/a$. Then, the circumference is given by

$$C = 2\pi a \sin\theta = 2\pi a \sin\frac{r}{a} \tag{4}$$

Then, the ratio is given by

$$\frac{C}{r} = 2\pi \frac{\sin\frac{r}{a}}{\frac{r}{a}} \tag{5}$$

As $\frac{\sin\theta}{\theta} < 1$, we see that the ratio is smaller than 2π. Also, it is interesting that there is a maximum value for the circumference. It is $2\pi a$. See Fig. 3. This is achieved when $\frac{r}{a} = \pi/2$. That is $r = \pi a/2$. Notice also that the circumference decreases as r increases if r is bigger than $\pi a/2$. What is the area of a circle with radius r? See Fig. 4. The area of "stripe" is given by $C\,dr$. We need to integrate this as follows.

$$A = \int_0^r 2\pi a \sin\frac{r}{a} dr = 2\pi a^2 (1 - \cos\frac{r}{a}) \tag{6}$$

Similarly as before, this value is smaller than $4\pi r^2$. (**Problem 1.** What is the maximum A can have? This should be equal to the surface area of 2-sphere with radius a. Thus, check that your answer is correct.)

If a is bigger, it means that the sphere is less curved. Think of the Earth. As a is very big, we don't usually notice that the Earth is not flat, but actually round. So, when a is big, (4) and (6) should reduce to our usual flat formula $C = 2\pi r$, and $A = 4\pi r^2$. (**Problem 2.** Show this using Taylor series!)

Having talked about 2-sphere, let's talk about 3-sphere. Let's say again the radius of the 3-sphere is a. Then, a circle in this 3-sphere satisfies (4). What would be the surface area of a 2-sphere in the 3-sphere? It turns out[1]

$$A = 4\pi \left(a \sin \frac{r}{a} \right)^2 \tag{7}$$

An easy way of interpreting this formula is that you regard $a \sin \frac{r}{a}$ as the "effective" radius as in (4). In other words, 2π times the effective radius is the circumference and 4π times the square of the effective radius is the area of the sphere.

Problem 3. Calculate the volume of 3-ball of radius r inside 3-sphere using (7). Then, show that this reduces to the familiar $V = \frac{4}{3}\pi r^3$ in the limit a goes to infinity. What is the maximum volume a 3-ball can have?

Spheres are examples of constant curvature. Curvature denotes how much manifolds are curved, and because of the symmetry of spheres all the points on the sphere have the same curvature. The curvature turns out to be $1/a^2$ in our cases. Therefore, precisely speaking, spheres are examples of constant positive curvature.

There are examples of constant negative curvature. They are called "pseudo-spheres." For the pseudo-spheres with constant curvature $-1/a^2$, if we actually calculate the analogous formula for the formulas we had for spheres, all the sine functions in our formulas are replaced by sine hyperbolic functions. For example, for (4), we have

$$C = 2\pi a \sinh \frac{r}{a} \tag{8}$$

for (7), we have

$$A = 4\pi \left(a \sinh \frac{r}{a} \right)^2 \tag{9}$$

Actually, even though we will not show here, it turns out that there are infinitely many lines that are parallel to a certain line and pass a point. Also, the sum of the angles of the triangle is always less than 180°.

Problem 4. Show that the ratio of the circumference to the radius is always bigger than 2π for circles in 2 pseudo-sphere. Also show that in the

[1] We will later show this in our article "FRW metric."

flat limit (i.e. a goes to infinity), the ratio becomes 2π. Notice that in this limit, the curvature (i.e. $-1/a^2$) approaches 0.

Problem 5. Show that (9) can be as big as it can be. This shows that a 2 pseudo-sphere has an infinite area.

Let me conclude this article with a comment. In "Cosmological principle and my view on philosophy" I introduced the cosmological principle. Cosmological principle suggests that our Universe has a constant curvature. So, there are three possibilities for the geometry of the space of our Universe. 3-sphere, the Euclidean space $\mathbb{R}^3$ and 3-pseudo sphere. If our universe is 3-sphere, the size of our universe is finite. If our Universe is the Euclidean space or 3-pseudo sphere, the size of our universe is infinite. All these three possibilities for our Universe have been considered by cosmologists. We will talk more about these three possibilities in our later article "FRW metric."

Summary

- The inner product of $\vec{x} = (x_1, x_2, \cdots, x_n)$ and $\vec{y} = (y_1, y_2, \cdots, y_n)$, two vectors in the Euclidean space $\mathbb{R}^n$, is given by $\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.
- The inner product of $\vec{x} = (x_1, x_2, \cdots, x_n)$ and $\vec{y} = (y_1, y_2, \cdots, y_n)$, two vectors in the Minkowski space $\mathbb{R}^n$, is given by $\vec{x} \cdot \vec{y} = -x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.
- The Euclidean space and the Minkowski space are examples of manifolds with no curvature.
- In the Euclidean space, there is only one straight line that is parallel to another certain straight line and passes a certain point.
- On a sphere, which has a constant positive curvature, there is no such a parallel line.
- On a sphere, the sum of the angles of a triangle is always bigger than 180°.
- In the limit, the radius of a sphere becomes infinite, (i.e. the curvature of a sphere becomes zero) the sphere becomes a flat space.
- A pseudo-sphere has a constant negative curvature.
- In the limit, the curvature of a pseudo-sphere becomes zero, it becomes a flat space.