Pauli matrices and spinor

In this article, we will represent the angular momentum operator by matrices. To this end, recall that in the last article we asserted that $L_+|j,\ m>$ is proportional to $|j,\ m+1>$ while $L_-|j,\ m>$ is proportional to $|j,\ m-1>$. Now, let's find the explicit proportionality constant. First, we will define $|j,\ m>$s to be orthonormal as follows:

$$< j_1,\ m_1|j_2,\ m_2 >= \delta_{j_1 j_2}\delta_{m_1 m_2} \tag{1}$$

This is possible since $|j,\ m>$s are eigenvectors of Hermitian matrices (i.e. L^2 and L_Z.). Now, we have:

$$\begin{aligned} L_+|j,\ m> &= c|j,\ m+1> && (2)\\ < j,\ m|L_-L_+|j,\ m> &= < j,\ m+1|c^*c|j,\ m+1> \\ < j,\ m|L^2 - L_z^2 - \hbar L_z|j,\ m> &= |c|^2 \\ (j(j+1) - m^2 - m)\hbar^2 &= |c|^2 \\ |c| &= \hbar\sqrt{j(j+1) - m(m+1)} && (3) \end{aligned}$$

where from the first line to the second line, we used $(L_+)^\dagger = L_-$ and where from the second line to the third line, we used (18) in last article. Without loss of generality, one can choose c to be real. In other words, $c = |c|$. This is possible by performing global gauge transformation upon the eigenvector $|j,\ m+1>$ in (2). Therefore, we conclude:

$$L_+|j,\ m>= \hbar\sqrt{j(j+1) - m(m+1)}|j,\ m+1> \tag{4}$$

Similarly, one can show (**Problem 1.**)

$$L_-|j,\ m>= \hbar\sqrt{j(j+1) - m(m-1)}|j,\ m-1> \tag{5}$$

Now, let's find the matrix representation of angular momentum, when $j = 1/2$. We know that its vector space is 2-dimensional since we have $|1/2,\ 1/2>$ and $|1/2,\ -1/2>$. We can represent this by:

$$|1/2,\ 1/2>= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1/2,\ -1/2>= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6}$$

The first one is called "spin up," and the other is called "spin down." A general state in this vector space can be represented by a linear combination of these two vectors as follows:

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix} \tag{7}$$

This representation called "spinor" is used to express the state vector of spin-1/2 particle such as electron. Now, observe the followings:

$$L_z|1/2,\ 1/2>= \frac{\hbar}{2}|1/2,\ 1/2>, \qquad L_z|1/2,\ \ -1/2>= -\frac{\hbar}{2}|1/2,\ \ -1/2> \tag{8}$$

implies:

$$L_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

Similarly,

$$L_+|1/2,\ 1/2>= 0, \qquad L_+|1/2,\ \ -1/2>= \hbar|1/2, 1/2> \tag{10}$$

implies:

$$L_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{11}$$

$$L_-|1/2,\ 1/2>= \hbar|1/2, -1/2>, \qquad L_-|1/2,\ \ -1/2>= 0 \tag{12}$$

implies:

$$L_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{13}$$

Using, $L_x = (L_+ + L_-)/2$, and $L_y = (L_+ - L_-)/(2i)$, we obtain

$$L_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad L_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{14}$$

If we define σ by $\vec{L} = (\hbar/2)\vec{\sigma}$, we conclude:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{15}$$

These are called "Pauli matrices."

(6) are eigenvectors of L_z. What are the eigenvectors of L_x and L_y? Using (14), one can check that

$$\alpha_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \beta_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{16}$$

are eigenvectors of L_x with eigenvalues $\hbar/2$, $-\hbar/2$ respectively, and

$$\alpha_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \beta_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{17}$$

are eigenvectors of L_x with eigenvalues $\hbar/2$, $-\hbar/2$ respectively.

All the constructions in this article were based on 3 spatial dimensional cases; we used x, y, z and p_x, p_y, p_z. We showed that this led to spinor having two components. In 1928, Dirac found "Dirac matrices" which are 4-dimensional ("4d") analogs of Pauli matrices. There, the spinor has four components. In 10d or 11d, in which string theory and M-theory live respectively, the spinor has thirty-two components. All this would be interesting to talk about more in details, but this is out of scope for this series unfortunately.

Finally, let me conclude this article with a comment. One can take similar steps as in $j = 1/2$ case to calculate the angular momentum matrices for other j. (Of course, we are talking about 3d case as before.) For example, for $j = 1$, we obtain:

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{18}$$

$$L_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{19}$$

$$L_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{20}$$

Problem 2. Suppose, an electron is in an eigenstate of L_z with eigenvalue $-\hbar/2$. What is the probability that its L_x will be $\hbar/2$ if you measure it?

Problem 3. Check (18), (19) and (20).

Problem 4. So far we have considered the angular momentum along x, y or z axis. More generally, we can consider the angular momentum along an arbitrary direction $\hat{r}$ given as follows using spherical coordinate system:

$$\hat{r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k} \tag{21}$$

Show that the eigenspinors of angular momentum along $\hat{r}$ with eigenvalues respectively $\hbar/2$ and $-\hbar/2$ are given as follows:

$$\alpha_{\hat{r}} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}, \quad \beta_{\hat{r}} = \begin{pmatrix} e^{-i\phi}\sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix} \tag{22}$$

For example, for L_z we have $\theta = 0$ and $\phi = 0$, so we obtain (6). For L_x, we have $\theta = \pi/2$ and $\phi = 0$, so we obtain (16). For L_y, we have $\theta = \pi/2$ and $\phi = \pi/2$, so we obtain (17) upto overall phase. (If $\vec{v}$ is a normalized eigenvector, $e^{i\lambda}\vec{v}$ is also a normalized eigenvector with the same eigenvalue. Here $e^{i\lambda}$ is the overall phase with λ being real.)(Hint[1])

Summary

- $|1/2,\ 1/2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1/2,\ -1/2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- The first one is called "spin up" and the other one is called "spin down."

[1]The angular momentum along the direcion $\hat{r}$ is given by $\hat{L}\cdot\hat{r}$ where $\hat{L} = L_x\hat{i} + L_y\hat{j} + L_z\hat{k}$.

- These two-dimensional space is called “spinor.”
- $\vec{L} = (\hbar/2)\vec{\sigma}$. $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$