The Aharonov-Bohm effect

In this article, we will consider how the presence of magnetic field affects the wave function of a charged particle. To this end, recall that in such a case Hamiltonian was given as follows:

$$H = \frac{(\vec{p} - q\vec{A})^2}{2m} + q\phi \tag{1}$$

Translating this into quantum mechanics, we have:

$$E\psi = \left(\frac{(-i\hbar\nabla - q\vec{A})^2}{2m} + q\phi \right) \psi \tag{2}$$

Now, let's say a charged particle with the wave function ψ travels from the point $\vec{r}_i$ to the point $\vec{r}$ in the presence of magnetic field somewhere near the path. Then, how can we obtain $\psi(\vec{r})$? To this end, we will say that $\psi_0(\vec{r})$ is the wave function solution of the same Hamiltonian without the presence of magnetic field near the path. Then, we will try to express $\psi(\vec{r})$ in terms of $\psi_0(\vec{r})$ and $\vec{A}$. In other words,

$$E\psi_0 = \left(\frac{(-i\hbar\nabla)^2}{2m} + q\phi \right) \psi_0 \tag{3}$$

Now, we claim:

$$\psi(\vec{r}) = e^{if} \psi_0(\vec{r}) \tag{4}$$

where

$$f = \frac{q}{\hbar} \int_{r_i}^{r} \vec{A}(\vec{r}') \cdot d\vec{r}' \tag{5}$$

where the line integration is done along the path that the particle actually takes.

To prove this, we have to show (**Problem 1.**):

$$\left(-i\hbar\nabla - q\vec{A} \right) \psi = -i\hbar e^{if} \nabla \psi_0 \tag{6}$$

Doing exactly the same thing again, we get:

$$\left(-i\hbar\nabla - q\vec{A} \right)^2 \psi = -e^{if} (-i\hbar\nabla)^2 \psi_0 \tag{7}$$

Plugging this into (2) and cancelling e^{if} factors, we obtain (3). This completes the proof. Now comes the question. Could we confirm this phase factor e^{if} experimentally? See Fig.1. Here, we are performing double slit experiment using electrons, and path 1 and path 2 are drawn, which have the same length. We could have chosen to draw other paths that have different lengths, but argument that we will present can carry over to such cases easily, and for simplicity we consider the same ones.

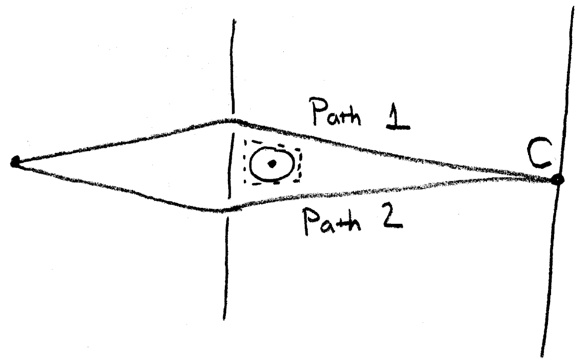

Figure 1: Double slit experiment with magnetic field

As you see, the magnetic field is present inside the dotted region. Outside the dotted region, the magnetic field is not present, including on the path 1 and path 2. Therefore, at first glance, the presence of magnetic field in dotted region may not seem to affect the wave function or the phase of the electron, as long as the magnetic field is not present on the paths; it may seem that there would be a constructive interference at the point C, because path 1 and path 2 have the same length.

However, this is not true as explicit calculations can show. If the additional phase an electron gains by going through the path 1 is f_1 and the one the path 2 is f_2, we have:

$$f_1 = \frac{-e}{\hbar} \int_{\text{path 1}} \vec{A} \cdot d\vec{r} \tag{8}$$

where $-e$ is the charge of the electron, and we have a similar expression for f_2. Now, what we are interested is neither f_1 nor f_2 themselves, but their difference as it is the only thing that matters when one tries to find whether we have constructive interference or destructive interference. Therefore, we should consider:

$$f_2 - f_1 = \frac{-e}{\hbar} \int_{\text{path 2}} \vec{A} \cdot d\vec{r} - \frac{-e}{\hbar} \int_{\text{path 1}} \vec{A} \cdot d\vec{r} = \frac{-e}{\hbar} \int_{\text{path 2}-\text{path 1}} \vec{A} \cdot d\vec{r} \tag{9}$$

Now, notice that $\nabla \times \vec{A} = \vec{B}$. Therefore, as we have seen in "Electric potential and vector potential," we have:

$$\Phi_B = \int_{\text{path 2}-\text{path 1}} \vec{A} \cdot d\vec{r} = \int_M \vec{B} \cdot d\vec{A} \tag{10}$$

where M is the region bounded by path 1 and path 2. In other words, Φ_B is the magnetic flux in this region. Notice that Φ_B is not zero, as magnetic flux is present inside the dotted region which is inside the region M. Therefore, depending on Φ_B we can either have constructive interference or destructive interference, even though the magnetic field is zero on the actual paths (i.e. path 1 and path 2) that electrons take. So, this is really bizarre. This is called "Aharonov-Bohm effect" and was experimentally confirmed.

Problem 2. What value does Φ_B have to be to have constructive interference at point C? How about destructive interference?

Summary

- Compared to the case in the absence of vector potential, the wave function gains an extra phase proportional to $\int \vec{A} \cdot d\vec{r}$.
- This leads to Aharonov-Bohm effect, in which a magnetic field can affect charged particles despite the absence of the magnetic field in the actual path of the charged particles.