Ashtekar variables

1 Introduction

In this article, we will review Ashtekar variables closely following *Quantum Gravity* by Rovelli. This step is important because our construction of our "newer" variables will closely follow this construction. The convention in this article is as follows. The Lorentz indices i, j, k take 1, 2, and 3 for their values, and the Lorentz indices I, J, K take 0, 1, 2, and 3 as their values. Also, we use Greek letters for spacetime indices, and a, b, c for space indices.

2 Self-dual and anti-self-dual

Remember that in four-dimensions, if you take a Hodge dual of an n-form, you get a $(4-n)$-form. Therefore, if you take a Hodge dual of an n-form twice, you will get an n-form again. Actually, it turns out that so-obtained n-form is proportional to the original n-form. The proportionality constant is 1 or -1, depending on the number of dimensions, signature of the metric and n. In case of 2-forms, the proportionality constant is -1. For examples,

$$* * (e^2 \wedge e^3) = *(e^0 \wedge e^1) = -e^2 \wedge e^3 \tag{1}$$

$$* * (e^0 \wedge e^1) = *(-e^2 \wedge e^3) = -e^0 \wedge e^1 \tag{2}$$

So, for any two-forms, we have $** = -1$. Thus, the eigenvalues of $*$ are i and $-i$. The eigenvector with eigenvalue i is called "self-dual" 2-form, and the eigenvector with eigenvalue $-i$ is called "self-dual." Thus, any two-form can be written into as a sum of anti-self-dual part and self-dual part. If we have a two-form A, the self-dual projection operator P^+ and the anti-self-dual projection operator P^- are given by

$$P^+ = \frac{1-i*}{2}, \qquad P^- = \frac{1+i*}{2} \tag{3}$$

Then, a 2-form T can be written as

$$T = P^+T + P^-T \tag{4}$$

where P^+T is an self-dual 2-form and P^- is a anti-self-dual 2-form.

Problem 1. Check that P^+T has indeed an eigenvalue i with respect to the $*$ operator. i.e. $*P^+T = iP^-T$. Similarly, you can check $*P^-T = -iP^-T$.

Problem 2. Check that $(P^{\pm})^2 = P^{\pm}$. This implies that once you project a 2-form into its self-dual part, if you project so-obtained self-dual part once again to its self-dual part, it will remain same. A similar interpretation can be made about anti-self-dual projection.

Problem 3. Check that $P^+P^- = 0$. This implies that if you project a 2-form into anti-self-dual part, there will be no self-dual component for the resulting anti-self-dual 2-form.

Anyhow, we can easily see that the basis for self-dual 2-form is given by

$$\Sigma^1 = e^2 \wedge e^3 - ie^0 \wedge e^1 \tag{5}$$

$$\Sigma^2 = e^3 \wedge e^1 - ie^0 \wedge e^2 \tag{6}$$

$$\Sigma^3 = e^1 \wedge e^2 - ie^0 \wedge e^3 \tag{7}$$

In other words,

$$\Sigma^i = \frac{1}{2}\tilde{\epsilon}^i_{jk} e^j \wedge e^k - ie^0 \wedge e^i \tag{8}$$

3 Plebanski formalism

Now, let's define the self-dual complex $SO(3)$ connection as follows:

$$A^1 = \omega^{32} + i\omega^{01} \tag{9}$$

$$A^2 = \omega^{13} + i\omega^{02} \tag{10}$$

$$A^3 = \omega^{21} + i\omega^{03} \tag{11}$$

As an aside, in the presence of Immirzi parameter, we have:

$$A^1 = -\omega^{32} + \gamma\omega^{01} \tag{12}$$

and similarly for the other components, where γ is Immirzi parameter which is real.

Then, the claim is that the Einstein-Hilbert action is given by

$$S = \frac{-2i}{16\pi G}\int \Sigma_i \wedge F^i \tag{13}$$

where F is the curvature corresponding to A.

Let's prove this. If you recall the lesson of the article "Gauge transformation in dreibein," the curvature should be given by

$$F^1 = dA^1 + A^2 \wedge A^3 \tag{14}$$

$$F^2 = dA^2 + A^3 \wedge A^1 \tag{15}$$

$$F^3 = dA^3 + A^1 \wedge A^2 \tag{16}$$

Let's calculate them explicitly.

$$
\begin{aligned}
F^1 &= dA^1 + A^2 \wedge A^3 && (17)\\
&= d\omega^{32} + \omega^{13} \wedge \omega^{21} - \omega^{02} \wedge \omega^{03} + i(d\omega^{01} + \omega^{02} \wedge \omega^{21} + \omega^{13} \wedge \omega^{03}) && (18)\\
&= d\omega^{32} + \omega^{31} \wedge \omega^{12} - \omega^{30} \wedge \omega^{02} + i(d\omega^{01} + \omega^{02} \wedge \omega^{21} + \omega^{03} \wedge \omega^{31}) && (19)\\
&= d\omega^{32} + \omega^{3}{}_{1} \wedge \omega^{12} + \omega^{3}{}_{0} \wedge \omega^{02} + i(d\omega^{01} + \omega^{0}{}_{2} \wedge \omega^{21} + \omega^{0}{}_{3} \wedge \omega^{31}) && (20)\\
&= R^{32} + iR^{01} && (21)
\end{aligned}
$$

Similarly, we have:

$$F^i = \frac{1}{2}\tilde{\epsilon}^{i}{}_{lm} R^{ml} + iR^{0i} \qquad (22)$$

We see here that the curvature is self-dual if the connection is self-dual.

Now, let's calculate

$$
\begin{aligned}
\Sigma^i \wedge F^i &= \frac{i}{2}\epsilon_{ijk} e^j \wedge e^k \wedge R^{0i} + \frac{i}{2}\epsilon_{ilm} e^0 \wedge e^i \wedge R^{lm} && (23)\\
&\quad -\frac{1}{2} e^l \wedge e^m \wedge R^{lm} + e^0 \wedge e^i \wedge R^{0i} && (24)\\
&= \frac{i}{4}\epsilon_{IJKL} e^I \wedge e^J \wedge R^{KL} - \frac{1}{2} e_I \wedge e^J \wedge R^{I}{}_{J} && (25)
\end{aligned}
$$

The last term in (25) vanishes due to the Bianchi identity. Thus, we conclude (13).

4 Ashtekar formalism

Consider a solution $(e^I_\mu(x), A^I_\mu(x))$ of the Einstein equations. Choose a 3d surface σ : $\overrightarrow{\tau} = (\tau^a) \to x^\mu(\overrightarrow{\tau})$ without boundaries in the coordinate space. The four-dimensional forms A^I, Σ^I and e^I induce the following three-dimensional forms.

$$A^I(\overrightarrow{\tau}) = A^I_a(\overrightarrow{\tau}) d\tau^a \qquad (26)$$

$$\Sigma_I(\overrightarrow{\tau}) = \frac{1}{2}\Sigma_{Iab}(\overrightarrow{\tau}) d\tau^a \wedge d\tau^b \qquad (27)$$

$$e^I(\overrightarrow{\tau}) = e^I_a(\overrightarrow{\tau}) d\tau^a \qquad (28)$$

Let's write $e^I = (e^0, e^i)$. Let's momentarily choose a gauge in which

$$e^0 = 0 \qquad (29)$$

to obtain the Ashtekar variable E^i in terms of the dreibein part of the vierbein.

In this gauge, from (5), (6), and (7) we have:

$$
\begin{aligned}
\Sigma^1 &= e^2 \wedge e^3\\
\Sigma^2 &= e^3 \wedge e^1\\
\Sigma^3 &= e^1 \wedge e^2 \qquad (30)
\end{aligned}
$$

which is indeed the area two-form. Now, area operator can be re-expressed as

$$\begin{aligned} \Sigma^i &= E^i && (31)\\ \frac{1}{2}\Sigma^i_{bc}dx^b\wedge dx^c &= \frac{1}{2}E^{id}\tilde{\epsilon}_{dbc}dx^b\wedge dx^c && (32)\\ \Sigma^i_{bc} &= E^{id}\tilde{\epsilon}_{dbc} && (33)\end{aligned}$$

Multiplying $\tilde{\epsilon}^{abc}$ both sides, we obtain

$$E^{ia} = \frac{1}{2}\tilde{\epsilon}^{abc}\Sigma^i_{bc} \tag{34}$$

Furthermore, from

$$\Sigma_i = E_i = \frac{1}{2}\tilde{\epsilon}_{ijk}e^j\wedge e^k = \frac{1}{2}\Sigma^i_{bc}dx^b\wedge dx^c \tag{35}$$

we have

$$\Sigma_{ibc} = \tilde{\epsilon}_{ijk}e^j_b e^k_c \tag{36}$$

which, from (34) implies

$$E^a_i = \frac{1}{2}\tilde{\epsilon}^{abc}\tilde{\epsilon}_{ijk}e^j_b e^k_c \tag{37}$$

From this relation, we can also obtain

$$E^a_i = (\det e)e^a_i \tag{38}$$

Given this, we can now obtain the action in terms of Ashtekar variables.

$$\begin{aligned} S &= \frac{-2i}{16\pi G}\int\Sigma_i\wedge F^i = \frac{-2i}{16\pi G}\int\frac{1}{4}\Sigma_{i\mu\nu}F^i_{\rho\sigma}\tilde{\epsilon}^{\mu\nu\rho\sigma}d^4x\\ &= \frac{-2i}{16\pi G}\int\frac{1}{2}(\Sigma_{iab}F^i_{0c}+\Sigma_{i0a}F^i_{bc})\tilde{\epsilon}^{abc}d^4x\\ &= \frac{-i}{8\pi G}\int(E^c_i(\partial_0A^i_c-\partial_cA^i_0+\tilde{\epsilon}^i{}_{jk}A^j_0A^k_c)+2(\tilde{\epsilon}^i{}_{jk}e^j_ae^k_0+ie^0_0e^i_a+ie^0_ae^i_0)F_{ibc}\tilde{\epsilon}^{abc})d^4x\\ &= \frac{-i}{8\pi G}\int(E^c_i\dot{A}^i_c+A^i_0D_cE^c_i+2(\tilde{\epsilon}^i{}_{jk}e^j_ae^k_0+ie^0_0e^i_a)F_{ibc}\tilde{\epsilon}^{abc})d^4x && (39)\\ &= \frac{-i}{8\pi G}\int(E^c_i\dot{A}^i_c+\Lambda^i(D_cE^c_i)+N^b(E^a_iF^i_{ab})+\tilde{N}(\tilde{\epsilon}^{jk}{}_iE^a_jE^b_kF^i_{ab}))d^4x && (40)\end{aligned}$$

where from the third line to the fourth line we used integration by parts (i.e. the fact that the total derivatives do not contribute to the integration) and the gauge condition $e^0_a = 0$. From (39) to (40), we re-labeled the variables. (**Problem 1.** Derive (40) from (39) using (33), (36), (37) and (38.) From above action, it is easy to read off the following Poisson bracket

$$\{A^i_a(\overrightarrow{r}), E^b_j(\overrightarrow{r}')\} = (i)\delta^i_j\delta^b_a\delta^3(\overrightarrow{r},\overrightarrow{r}') \tag{41}$$

where we have set $8\pi G = 1$. Also, as an action is always an extremum, we must have:

$$0 = \frac{\delta S}{\delta\Lambda^i} = \frac{\delta S}{\delta N^b} = \frac{\delta S}{\delta\tilde{N}} \tag{42}$$

which implies following equations called "constraints."

$$
\begin{aligned}
D_c E_i^c &= 0 \\
E_i^a F_{ab}^i &= 0 \\
\tilde{\epsilon}^{jk}{}_i E_j^a E_k^b F_{ab}^i &= 0
\end{aligned} \tag{43}
$$

Notice how these constraints are derived. In the action, there are no time derivatives for $\Lambda^i, N^b, \tilde{N}$. So, we simply had (43). It means that $\Lambda^i, N^b, \tilde{N}$ can be interpreted as Lagrange multipliers.

The first equation is called "Gaussian constraint," the second equation is called "diffeomorphism constraint," and the third equation is called "Hamiltonian constraint" or "scalar constraint."

Now, from (40), let's find the Hamiltonian. Using

$$
S = \int L\, dt = \int (p\dot{q} - H) dt \tag{44}
$$

it is given by

$$
H = \frac{i}{8\pi G} \int (\Lambda^i (D_c E_i^c) + N^b (E_i^a F_{ab}^i) + N(\frac{1}{2} \tilde{\epsilon}^{jk}{}_i E_j^a E_k^b F_{ab}^i)) d^3x \tag{45}
$$

Notice that the total Hamiltonian is zero. This is expected from the following reason. Remember that in general relativity, the time coordinate (as well as space coordinate) has no intrinsic meaning, because it is just a label that we can freely choose to parametrize spacetime manifold. As Hamiltonian generates the time translation, which has no meaning, it should be zero.

From now on, we will denote the Lorentz space indices by vector symbols. Then, the above Hamiltonian can be written as

$$
H = \frac{1}{8\pi G} \Big(\mathcal{G}(\vec{\Lambda}) + \mathcal{D}(N^b) + \mathcal{H}(N) \Big) \tag{46}
$$

where

$$
\begin{aligned}
\mathcal{G}(\vec{\Lambda}) &= i \int d^3x \vec{\Lambda} \cdot D_c \vec{E}^c \\
\mathcal{D}(N^b) &= i \int d^3x N^b \vec{E}^a \cdot \vec{F}_{ab} \\
\mathcal{H}(\tilde{N}) &= i \int d^3x \tilde{N} \frac{1}{2} \vec{F}_{ab} \cdot (\vec{E}^a \times \vec{E}^b)
\end{aligned} \tag{47}
$$

Now, let's find the Poisson bracket of Ashtekar variables with the above three constraints. We get

$$
\{\vec{A}_a, \mathcal{G}(\vec{\Lambda})\} = D_a \vec{\Lambda} \tag{48}
$$

$$
\{\vec{E}_a, \mathcal{G}(\vec{\Lambda})\} = \vec{E}_a \times \vec{\Lambda} \tag{49}
$$

$$\{\vec{A}_a, \mathcal{D}(N^b)\} = \mathcal{L}_N A_a - \{A_a, \mathcal{G}(N_b \vec{A}^b)\} \tag{50}$$

$$\{\vec{E}^a, \mathcal{D}(N^b)\} = -\mathcal{L}_N E^a + \{E^a, \mathcal{G}(N_b \vec{A}^b)\} \tag{51}$$

$$\{\vec{A}_a, \mathcal{H}(\tilde{N})\} = \tilde{N} \vec{F}_{ab} \times \vec{E}^b \tag{52}$$

$$\{\vec{E}^a, \mathcal{H}(\tilde{N})\} = -\tilde{N} D_b(\vec{E}^a \times \vec{E}^b) \tag{53}$$

Now, notice from (48) and (49) that the Gaussian constraint generates the gauge freedom to choose dreibein and spin connection, which we learned in “Gauge transformations in dreibein.” We can see the Gaussian constraint from a slightly different point of view as well. If we denote the wave function by $\Psi(A^i_c)$, the gauge transformation means

$$\psi(A^i_c) = \psi(A^i_c + D_c \Lambda^i) \tag{54}$$

which implies

$$\int d^3x D_c \Lambda^i \frac{\delta \psi}{\delta A^i_c} = 0 \tag{55}$$

Using the fact that taking derivative with respect to A^i_c is the same thing as multiplying by its conjugate variable E^c_i, and using integration by parts, we get

$$\int d^3x \Lambda^i D_c E^c_i = 0 \tag{56}$$

Now, let’s move on to the diffeomorphism constraint.

$$\mathcal{D}'(N^b) = \mathcal{D}(N^b) - \mathcal{G}(N^b \vec{A}_b) \tag{57}$$

we have

$$\{\vec{A}_a, \mathcal{D}'(N^b)\} = \mathcal{L}_N A_a, \qquad \{\vec{E}^a, \mathcal{D}'(N^b)\} = -\mathcal{L}_N E^a \tag{58}$$

We see that $\mathcal{D}'$ generates diffeomorphism. As both the Gaussian constraint and the diffeomorphism constraint are zero, they cannot generate physically meaningful evolution. This is indeed true as both the gauge transformations and the diffeomorphism of Ashtekar variables are not physically meaningful.

Now, let’s calculate the Poisson brackets between the constraints.

$$\{\mathcal{G}(\vec{\Lambda}_1), \mathcal{G}(\vec{\Lambda}_2)\} = \mathcal{G}(\vec{\Lambda}_1 \times \vec{\Lambda}_2) \tag{59}$$

$$\{\mathcal{G}(\vec{\Lambda}), \mathcal{D}(N^b)\} = 0 \qquad \text{i.e.} \qquad \{\mathcal{G}(\vec{\Lambda}), \mathcal{D}'(N^b) = -\mathcal{G}(\mathcal{L}_N \vec{\Lambda})\} \tag{60}$$

$$\{\mathcal{G}(\vec{\Lambda}), \mathcal{H}(\tilde{N})\} = 0 \tag{61}$$

$$\{\mathcal{D}'(M^a), D'(N^b)\} = D'(\mathcal{L}_M N^b) \tag{62}$$

$$\{\mathcal{D}'(N^b), \mathcal{H}(\tilde{M})\} = \mathcal{H}(\mathcal{L}_N \tilde{M}) \tag{63}$$

$$\{\mathcal{H}(\tilde{N}), \mathcal{H}(\tilde{M})\} = \mathcal{D}(K^a) = \mathcal{D}'(K^a) + \mathcal{G}(K^a \vec{A}_a) \tag{64}$$

where

$$K^a = 2\vec{E}^a \cdot \vec{E}^b (\tilde{N} \partial_b \tilde{M} - \tilde{M} \partial_b \tilde{N}) \tag{65}$$

We see that the Poisson brackets of the constraints can be expressed in terms of the linear combinations of the constraints themselves. In other words, the constraint algebra is closed. This is an important condition, since Poisson brackets (i.e. commutators) of zeros should be always zero.

Summary

- The Hodge operator for a 2-form in 4-dimensions has i and $-i$ as the eigenvalue, as $*^2 = -1$. One is called "self-dual" and the other is called "anti-self-dual."
- The self-dual projection operator and the anti-self-dual projection operator satisfy

$$P^+ + P^- = 1, \qquad (P^\pm)^2 = P^\pm, \qquad P^+P^- = 0$$

- The curvature of self-dual connection is also self-dual.
- The Einstein-Hilbert action is proportional to $\Sigma_i \wedge F^i$.
- The Ashtekar variables A and E are conjugate to each other.
- The Hamiltonian is zero, and is given by the sum of Gaussian constraint, the diffeomorphism constraint and the Hamiltonian constraint.

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