Bose-Einstein condensate

According to Wikipedia, Bose-Einstein condensate is "a state of matter of a dilute gas of bosons cooled to temperatures very close to absolute zero." To explain Bose-Einstein condensate let's recall Bose-Einstein distribution. The total number of a boson is given by

$$N = \frac{1}{h^3} \int \frac{d^3p d^3q}{e^{-\mu/kT} e^{\epsilon/kT} - 1} \tag{1}$$

If the boson can be treated non-relativistically, we have $p = \sqrt{2m\epsilon}$. Introducing the fugacity $z \equiv e^{\mu/kT}$ and using $\beta = 1/kT$, $d^3p = 4\pi p^2\, dp$ the above formula becomes

$$\frac{N}{V} = \frac{2\pi(2m)^{3/2}}{h^3} \int_0^\infty \frac{\epsilon^{1/2} d\epsilon}{z^{-1} e^{\beta\epsilon} - 1} \tag{2}$$

Now, notice that if the left-hand side is fixed, the bigger β the bigger z. So, if we put N molecules of a boson gas in a container with volume V and make the gas as cold as possible, z will get bigger and bigger. However, notice that the denominator $z^{-1}e^{\beta\epsilon} - 1$ must not be negative, otherwise it simply doesn't make any sense. Therefore,

$$z < e^{\beta\epsilon} \tag{3}$$

So, when is the right-hand side minimum? When $\epsilon = 0$. Therefore, no matter how cold the gas of boson may be, 1 is the maximum value z can have. So, let's calculate the right-hand side of (2) in the limit z goes to 1. We have

$$\frac{2\pi(2m)^{3/2}}{h^3} \int_0^\infty \frac{\epsilon^{1/2} d\epsilon}{e^{\beta\epsilon} - 1} = \frac{2\pi(2mkT)^{3/2}}{h^3} \int_0^\infty \frac{\sqrt{x} dx}{e^x - 1} \tag{4}$$

It is easy to see that the x integration converges. Actually, it is known as

$$\zeta\left(\frac{3}{2}\right) = \int_0^\infty \frac{\sqrt{x} dx}{e^x - 1} = 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \frac{1}{4^{3/2}} + \cdots \approx 2.612 \tag{5}$$

Thus, if we have N molecules of the boson in volume V and lower the temperature, z will get bigger and bigger to approach 1, to match the right-hand side of (2) with the left-hand side of (2), but as the temperature gets lower, it eventually satisfies

$$\frac{N}{V} > \frac{2\pi(2mkT)^{3/2}}{h^3}\zeta\left(\frac{3}{2}\right) \tag{6}$$

and, thus, cannot match the left-hand side of (2). Thus, something bizarre must be happening. Where were we wrong? Actually, (2) is not correct. We used the integral, but this is actually an approximation of sum. Considering this, we can actually separate the sum of different modes to when $\epsilon = 0$ and when $\epsilon > 0$. Then, we have

$$N = V\frac{2\pi(2mkT)^{3/2}}{h^3}\int_0^\infty \frac{\epsilon^{1/2}d\epsilon}{z^{-1}e^{\beta\epsilon}-1} + \frac{1}{z^{-1}e^{\beta\cdot 0}-1} \tag{7}$$

In other words,

$$\frac{N}{V} = \frac{2\pi(2m)^{3/2}}{h^3}\int_0^\infty \frac{\epsilon^{1/2}d\epsilon}{z^{-1}e^{\beta\epsilon}-1} + \frac{z}{1-z} \tag{8}$$

when T is so small that z is close to 1, the number of excited state (i.e. $\epsilon \neq 0$) is given by (4) as follows

$$N_e = V\frac{2\pi(2mkT)^{3/2}}{h^3}\zeta\left(\frac{3}{2}\right) \tag{9}$$

Then, the rest of them, which we denote by N_0 is in $\epsilon = 0$ as follows

$$N_0 = N - N_e = N - V\frac{2\pi(2m)^{3/2}}{h^3}\zeta\left(\frac{3}{2}\right) = \frac{z}{1-z} \tag{10}$$

Thus, the fugacity is given by

$$z = \frac{N_0}{N_0+1} \approx 1 - \frac{1}{N_0} \tag{11}$$

which is indeed very close to 1.

So, why is there no Fermi-Dirac condensation? It is because only one particle can occupy $\epsilon = 0$ state for fermions due to Pauli's exclusion principle. Therefore, it never happens that many states occupy $\epsilon = 0$ as in the Bose-Einstein condensation.

Let me conclude this article with historical remark. In 1924, an Indian physicist, Satyendra Nath Bose derived Planck's radiation law by the essentially same method presented in our last article. As his article was not accepted at once for publication, he sent his manuscript to Albert Einstein. Recognizing the importance of this work, Einstein translated Bose's paper from English into German and submitted it in Bose's name to a prestigious German journal, which published it. Einstein further worked on Bose's idea and came up with the concepts of boson and Bose-Einstein condensate. Bose-Einstein condensate was first produced in 1995 by American physicists, Eric Cornell and Carl Wieman. Soon, a US-based German physicist Wolfgang Ketterle demonstrated important properties of Bose-Einstein condensate. These three physicists won Nobel Prize in 2001. As I do not know much about what the the important properties of Bose-Einstein condensate are, I will stop my article here. I am not an expert.

Summary

- Bose-Einstein condensation is a phenomenon that many molecules of a dilute gas of boson occupy the state with energy zero in temperatures very close to absolute zero. There can't be such a condensate for fermions as at most one molecule can occupy $\epsilon = 0$ state due to Pauli's exclusion principle.