Planck's law of blackbody radiation

Suppose photons are filled in a box with temperature T. How many photons are there with given frequency between f and $f + df$? This is the question we will answer in this article.

Let's first think about a single mode (i.e. state) with frequency f. By Planck's relation, a photon with frequency with f has energy hf. As a photon is a boson, it is possible that a multiple number of photons are within the same state. Therefore, the following values of energy is possible:

$$E = shf \tag{1}$$

where s is non-negative integer. It is the number of photons in a given mode. Now, what is the probability for having s photons in the mode? Remember the Boltzmann factor. The probability is proportional to $\exp(-shf/(kT))$. Then, the partition function is given as follows:

$$Z = \sum_{s=0}^{\infty} \exp(-shf/(kT)) = \frac{1}{1 - \exp(-hf/(kT))} \tag{2}$$

If you don't know how to derive this, please read our earlier article, "the sum of the geometric series." Therefore, the probability that there are s photons in this mode is given as follows:

$$P(s) = \frac{\exp(-shf/kT)}{Z} \tag{3}$$

Given this, what is the expectation value for the number of photons in this mode? We have:

$$\langle s \rangle = \sum_{s=0}^{\infty} sP(s) = Z^{-1} \sum_{s=0}^{\infty} s \exp(-shf/(kT)) \tag{4}$$

Now, let's define $y \equiv hf/(kT)$, for a calculational simplicity. Then, the summation on the right hand side can be re-expressed as follows:

$$\begin{aligned} \sum_{s=0}^{\infty} s \exp(-sy) &= -\frac{d}{dy} \sum_{s=0}^{\infty} \exp(-sy) \\ &= -\frac{dZ}{dy} = -\frac{d}{dy}\left(\frac{1}{1-\exp(-y)}\right) = \frac{\exp(-y)}{(1-\exp(-y))^2} \end{aligned} \tag{5}$$

Plugging this back to (4), we obtain:

$$\langle s \rangle = \frac{1}{\exp(hf/(kT)) - 1} \tag{6}$$

We now turn our attention to the density of states. How many states are there between f and $f + df$? Remember that the corresponding momentum is given by

$$p = \frac{E}{c} = \frac{hf}{c} \tag{7}$$

Therefore, naively, we must have following number of states:

$$\frac{(4\pi p^2 dp)}{h^3} V = \frac{4\pi f^2 df}{c^3} V \tag{8}$$

However, there are two polarizations for each state of photon. Therefore, the above formula should be modified as follows:

$$2\frac{4\pi f^2 df}{c^3} V \tag{9}$$

Multiplying (4) by this number, we conclude that between f and $f + df$, there are following number of photons:

$$\frac{8\pi f^2 df}{c^3} \frac{V}{\exp(hf/kT) - 1} \tag{10}$$

In fact, it is known that our universe's temperature is approximately 2.73 Kelvin. It was first found out by an accident discovery that the universe is filled with corresponding frequency of photons. (There are hardly any photons for frequency that satisfies $hf >> kT$, as the denominator of (10) would be too small. However, for a very small f the numerator would be too small. So, there are certain ranges for frequency in which there are many numbers of photons. This corresponds to hf is in order of kT.). Since energy of photons with frequency f is given by hf multiplied by the number of photons (i.e. (10)), We can say that du, the total energy density of photons with frequency between f and $f + df$ is given as follows, :

$$du = \frac{8\pi h f^3 df}{c^3(\exp(hf/kT) - 1)} \tag{11}$$

Now, another question. How can we relate this to the energy, due to the same frequency range of photons, radiating from the body? In other words, we want to obtain a formula for black body radiation. It turns out that the energy emitted during time t concerning the photon frequency between f and $f + Df$ is given by:

$$duAt\frac{c}{4} \tag{12}$$

where A is the area of the body. For the derivation, see the appendix at the very end of this article. Then the energy emitted per unit time, per unit area for the frequency of photons concerned between f and $f + df$ is given by

$$du\frac{c}{4} = \frac{2\pi A h f^3 df}{c^2(\exp(hf/kT) - 1)} \tag{13}$$

This is known as "Planck's law of blackbody radiation."

If you want to calculate the total energy emitted per unit time, per unit area, you can just integrate the above formula as follows:

$$\int du\frac{c}{4} = \int_0^\infty \frac{2\pi A h f^3 df}{c^2(\exp(hf/kT) - 1)} = \frac{2\pi^5 k^4}{15c^2 h^3} T^4 \tag{14}$$

(**Problem 1.** Using $\int_0^\infty x^3 dx/(e^x - 1) = \pi^4/15$ derive the above equation. Hint[1])

So, the total energy emitted is proportional to the fourth power of temperature. This is known as "Stefan-Boltzmann law." Notice where the number four comes from. It comes from the fact that our world, which we live in, has three spatial dimensions, and four is three plus one.

Let us conclude this article with some historical remarks. We have seen that the energy of a single mode is given by:

$$\langle s \rangle hf = \frac{hf}{\exp(hf/(kT)) - 1} \tag{15}$$

However, equipartition theorem, a key theorem in classical statistical mechanics (Remember our earlier article "Kinetic theory of gases"), predicts that the energy of a single mode is given by kT since there are two degrees of freedom: potential energy and kinetic energy. (Notice that the degree of freedom for potential energy was absent for molecules moving in a box. Photons are different, since it is regarded as harmonic oscillators classically.) One can check that this agrees with (15) in the limit $hf << kT$, by Taylor-expanding the exponential function in the denominator. Nevertheless, for $hf >> kT$, it clearly deviates from the classical value kT. This is very important because it makes (14) converged, as the integrand for high f is suppressed by big denominator $\exp(hf/kT) - 1$. On the other hand, if the value kT is used for (15), (14) would have been:

$$\int_0^\infty \frac{2\pi AkTf^2 df}{c^2} \tag{16}$$

which is infinite due to the divergence of the integrand for high f. This puzzle is called "ultraviolet catastrophe." In physics, ultraviolet means high frequency or, equivalently, short wavelength. Apparently, Planck's law of blackbody radiation solved this problem. Another interesting historical fact is that Planck first came up with the law to fit the experimental data, then later came up with the derivation.

Summary

- The average number of photons in a given mode of frequency f is given by

  $$\langle s \rangle = \frac{1}{e^{hf/(kT)} - 1}$$

  . Therefore, the average energy of photons in a given mode of frequency f is given by

  $$\langle s \rangle hf = \frac{hf}{e^{hf/(kT)} - 1}$$

  .

- The number of states between f and $f + df$ is proportional to $f^2 df$.

[1] let $x = hf/kT$

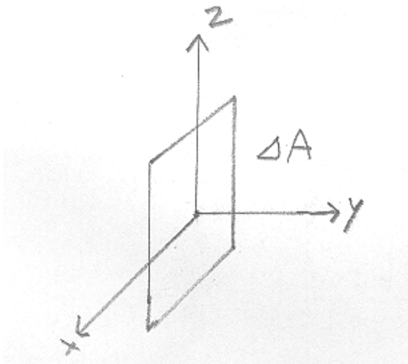

Figure 1: Small hole

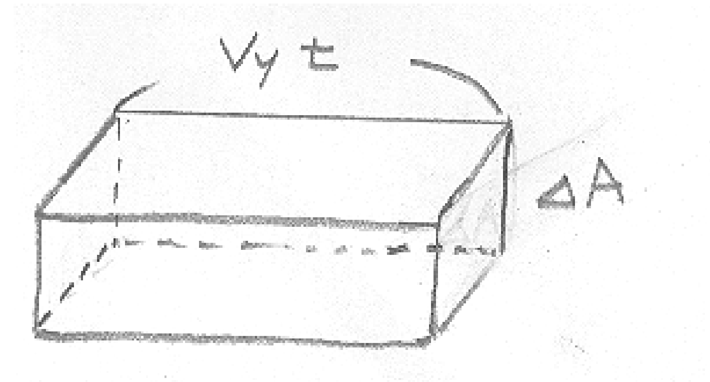

Figure 2: Photons coming out

- All together, the Planck radiation spectrum is proportional to

$$\frac{f^3 df}{e^{hf/(kT)} - 1}$$

- The total energy of blackbody radiation emitted per unit time, per unit area is proportional to T^4 where T is the blackbody temperature.

Appendix

Let's say that a box has a small hole with area ΔA, and it is situated in $x-z$ plane. See Fig.1. Photons randomly moving will accidentally come out of this hole. If the moving direction of certain photons is parallel to the hole, (i.e. if they are not approaching the hole, and just passing by) they won't come out. In other words, if the y component of the velocity of photon is zero, it won't come out. Also, if the photons are moving away from the hole, they won't come out either. (i.e. if the y component of the velocity of certain photons is negative) This shows that all that concerned is the y component of the velocity of the photons. Now, see Fig.2. If a photon has v_y for the y component of the velocity and during the time t, if it happens to be inside the rectangular parallelepiped drawn in the figure, it will come out. Thus, the energy that comes out during time t is the energy inside the rectangular parallelepiped, which is given by $du\Delta A v_y t$. Now, we need to calculate the average of v_y. See Fig.3. We drew a half-sphere with radius 1. We have $v_y = c\cos\theta$ and the probability that v_y will have a value between $c\cos\theta$ and $c\cos(\theta + d\theta)$ is proportional to the shaded region. As the total area of the sphere is 4π, the average of v_y is given as follows:

$$\langle v_y \rangle = \int_0^{\pi/2} c\cos\theta \left(\frac{2\pi \sin\theta d\theta}{4\pi} \right) = \frac{c}{4} \tag{17}$$

where the range for θ is chosen so that v_y is positive. Therefore, during time t, the energy emitted is given by:

$$du\Delta A t\frac{c}{4} \tag{18}$$

This is exactly the formula we wanted to derive.

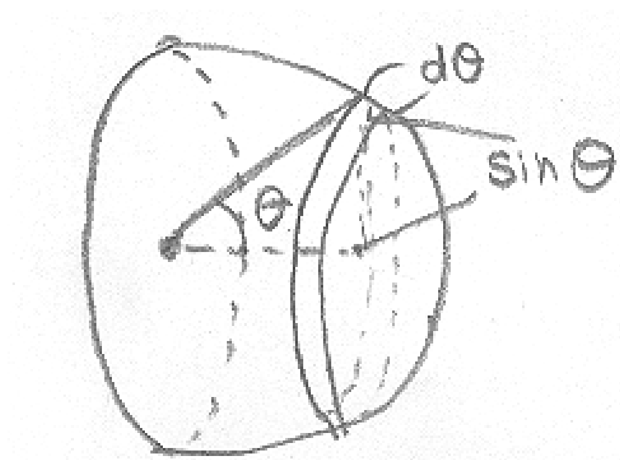

Figure 3: Calculation of average of v_y