Bohr model

In our earlier article "Rydberg formula," we briefly mentioned that Bohr came up with an explanation for Rydberg formula. In this article, we explain his model in more detail.

In 1911 Rutherford experimentally found out that an atom consists of a very tiny center called "nucleus" and electrons that orbit around it. As the nucleus is positively charged and electrons are negatively charged, there are Coulomb attractions between them, which make electrons orbit around the nucleus.

However, there is a serious problem with this model; it suggests that electrons in atoms are accelerating, since circular motion (i.e. orbiting around the center) is an example of acceleration. According to Maxwell's electrodynamics, any electrically charged particle accelerating loses energy as it emits electromagnetic wave (i.e. light). Therefore, the electrons orbiting around the nucleus should lose energy and eventually fall down to the nucleus within roughly 10^{-10} second according to a calculation. However, we know that atoms live longer than 10^{-10} second, let alone several years. If it didn't, the atoms in your body would decay quickly, so nothing would be left out of you.

To solve this problem, Bohr suggested that electrons in an atom have certain orbits, and energy is released or absorbed in form of light (i.e. photons) only when its orbits change. Of course, this violates the classical picture that electrons orbiting should always emit light, yet Bohr was bold. In particular, if the electron is in the lowest energy orbit, it cannot emit light to fall into a lower energy orbit, since there is no lower energy orbit. Therefore, atoms do not decay. Furthermore, he argued that in case of the hydrogen atom, the angular momentum of an electron orbiting around nucleus is given by positive integer multiples of reduced Planck constant $\hbar$ (i.e. Planck constant divided by 2π). For example, if an electron is in nth orbit, it has $n\hbar = nh/(2\pi)$ as its angular momentum. Then, if one carefully calculates, this condition implies that the electron's energy is inversely proportional to n^2, as mentioned in the last article. In particular, he successfully derived the right value for Rydberg constant.

Let's explicitly check this. The Coulomb force between the nucleus and the electron in an hydrogen atom gives the centripetal force of the electron. The nucleus of the hydrogen atom consists of a single proton with the electric charge e, while an electron has the electric charge $-e$. Therefore, we have:

$$\frac{mv^2}{r} = \frac{ke^2}{r^2} \tag{1}$$

where m is the mass of the electron (precisely speaking, the reduced mass of the electron and the nucleus of the hydrogen atom), v the orbiting speed, r the radius of the orbit, k Coulomb's constant. Solving the above equation, we get:

$$v = \sqrt{\frac{ke^2}{mr}} \tag{2}$$

As the total energy is given by the sum of kinetic energy and the potential energy, we have:

$$E = \frac{1}{2}mv^2 - \frac{ke^2}{r} = -\frac{ke^2}{2r} \tag{3}$$

Now comes Bohr's condition. The angular momentum is given as follows:

$$mvr = n\hbar \tag{4}$$

Plugging (2) to the above equation, we get:

$$\sqrt{ke^2mr} = n\hbar \tag{5}$$

Therefore, we conclude:

$$r = \frac{n^2\hbar^2}{ke^2m} \tag{6}$$

Plugging this to (3), we obtain:

$$E = -\frac{(ke^2)^2m}{2\hbar^2n^2} \tag{7}$$

Therefore, we see that the energy of the electron can have only above values. Now, if an electron falls from jth state to ith state the energy released is given as follows:

$$\Delta E = \frac{(ke^2)^2m}{2\hbar^2}\left(\frac{1}{i^2} - \frac{1}{j^2}\right) \tag{8}$$

This energy is released in form of a photon. This should equal hf where f is the frequency of the released photon. Therefore, we get:

$$f = \frac{(ke^2)^2m}{4\pi\hbar^3}\left(\frac{1}{i^2} - \frac{1}{j^2}\right) \tag{9}$$

As a photon with wavelength λ has c/λ as frequency, we have:

$$\frac{1}{\lambda} = \frac{(ke^2)^2m}{4\pi c\hbar^3}\left(\frac{1}{i^2} - \frac{1}{j^2}\right) \tag{10}$$

So, we obtained Rydberg constant as advertised!

Bohr also showed that his model agrees with classical picture (i.e. Maxwell's electrodynamics), when n is large. According to Maxwell's electrodynamics, the frequency of the photons emitted from orbiting electron should be the orbiting frequency. When an electron in nth orbit falls into $n-1$th orbit for a large n, one can show that the frequency of the photon thus released is roughly equal to the orbiting frequency of the electron; in the limit that n goes to infinity, the correspondence becomes exact.

Let's check this. The frequency for photon that is released when an electron falls from nth state to $n-1$th state can be simply obtained by plugging $i=n-1$, $j=n$ to (9). This yields:

$$f = \frac{(ke^2)^2 m}{4\pi\hbar^3}\left(\frac{1}{(n-1)^2} - \frac{1}{n^2}\right) = \frac{(ke^2)^2 m}{4\pi\hbar^3}\frac{(2n-1)}{(n-1)^2 n^2} = \frac{(ke^2)^2 m}{2\pi\hbar^3}\frac{(n-1/2)}{(n-1)^2 n^2} \tag{11}$$

On the other hand, the time that takes an electron to orbit once is given by:

$$T = \frac{2\pi r}{v} = 2\pi\sqrt{\frac{m}{ke^2}}r^{3/2} \tag{12}$$

where we have used (2). By definition the orbiting frequency is the inverse of T. Therefore, we get:

$$f_{\text{orb}} = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{ke^2}{m}}\frac{1}{r^{3/2}} = \frac{(ke^2)^2 m}{2\pi\hbar^3}\frac{1}{n^3} \tag{13}$$

where in the last step we have used (6). Now compare the above formula with (11). (11) becomes the above formula in large n limit. If you are not sure check this together now. In other words:

$$\lim_{n\to\infty}\frac{f}{f_{\text{orb}}} = \left(\frac{n-1/2}{(n-1)^2 n^2}\right) / \left(\frac{1}{n^3}\right) = \frac{n(n-1/2)}{(n-1)^2} = 1 \tag{14}$$

Therefore, f is exactly equal to f_{orb} in large n limit as advertised!

The correspondence such as this that quantum theory (i.e. non-classical, and microscopic) reduces to the classical one under macroscopic limit (in our case, big n) is called "correspondence principle." After all, microscopic law should always be able to explain everything macroscopic law already can.

Even though Bohr's model played a very important role in the formulation of quantum mechanics, his model was not correct. In particular, the actual lowest energy orbit, obtained by solving Schrödinger's equation, has zero angular momentum, while Bohr model said that it had $h/2\pi$ angular momentum. (i.e. $n=1$) Moreover, correspondence principle is no longer important since an exact law (i.e. quantum mechanics) is found.

Summary

- Rutherford experimentally found out that an atom consists of a very tiny center called nucleus and electrons that orbit around it.

- However, according to Maxwell's electrodynamics, any charged particle accelerating loses energy by emitting light. Therefore, the electrons orbiting around the nucleus should lose energy, and eventually fall down to the nucleus in a very short amount of time.
- To remedy this situation, Bohr suggested that electrons have certain orbits, and if the electron is in the lowest energy orbit, it cannot emit light to further fall into the nucleus.
- He suggested that the orbits of the electrons in hydrogen atom satisfy

  $$mvr = n\hbar$$

  where n is a positive integer. In other words, the angular momentum of the electron is an integer multiplies of $\hbar$.
- From this condition, he derived that nth energy level in a hydrogen atom is in the form of

  $$E_n = -\frac{\text{something}}{n^2}$$

  The numerator (denoted here as something) he obtained agreed with experiments.
- When n is big, Bohr showed that the frequency of light emitted corresponds to the frequency of the electron orbiting, as classical electrodynamics predicts.