The CMB anisotropy analysis

In our earlier article "History of astronomy from the early 20th century to the early 21st century," we have sketched how astronomers perform the CMB anisotropy analysis. In this article, we will explain how they actually do so with real mathematics.

The CMB temperature depends on the direction we are looking at. Let's call this direction $\hat{n}$. Then, the CMB temperature is $T(\hat{n})$. Then, we define ΔT, the deviation from the mean temperature by

$$\Delta T(\hat{n}) = T(\hat{n}) - \bar{T} \tag{1}$$

where $\bar{T}$ is the CMB temperature averaged over all directions i.e.,

$$\bar{T} = \frac{1}{4\pi} \int d^2\hat{n}\, T(\hat{n}) \tag{2}$$

where $d^2\hat{n}$ denotes the standard 2-sphere measure.

Given this, we can express the temperature deviation $\Delta T(\hat{n})$ as a linear combination of spherical harmonics as follows.

$$\Delta T(\hat{n}) = \sum_{\ell m} a_{\ell m} Y_\ell^m(\hat{n}) \tag{3}$$

We can easily obtain a_{lm} from the orthogonality of the spherical harmonics as follows.

$$a_{\ell m} = \int d^2\hat{n}\, \Delta T(\hat{n}) Y_\ell^m(\hat{n}) \tag{4}$$

Of course $a_{\ell m}$ depends on our coordinate. What does not depend on our coordinate is the following

$$C_\ell \equiv \langle |a_{\ell m}|^2 \rangle = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} a_{\ell m} a_{\ell m}^* \tag{5}$$

However, the cosmological theories cannot predict "C_ℓ" precisely, but with some inherent uncertainties called "cosmic variance." What the cosmological theories can predict is the probability distribution of C_ℓ. If we had many universes, we could measure C_ℓ in each universe, calculate its average and standard deviation, and compare it with the average and the standard deviation of C_ℓ the cosmological theories predicts. However, we cannot do so, because we have only one universe to observe C_ℓ. Anyhow, the cosmic variance decreases for high ℓ, as there are many $|a_{\ell m}|^2$ to be averaged, namely, $(2\ell+1)$ one of them, in such a case.

We can express C_ℓ directly in terms of ΔT. First, note that $\Delta T(\hat{n})$ is a real number. Thus, we have

$$\Delta T(\hat{n}) = \sum_{\ell m} a^*_{\ell m} Y^{m*}_\ell(\hat{n}) \tag{6}$$

which implies

$$\Delta T(\hat{n})\Delta T(\hat{n}') = \sum_{\ell m} a_{\ell m} a^*_{\ell m} Y^m_\ell(\hat{n}) Y^{m*}_\ell(\hat{n}') = \sum_\ell C_\ell \left(\frac{2\ell+1}{4\pi}\right) P_\ell(\hat{n}\cdot\hat{n}') \tag{7}$$

where we used the addition theorem for spherical harmonics. Then, using the orthogonality of Legendre polynomials

$$\int d^2\hat{n}' P_\ell(\hat{n}\cdot\hat{n}') P_{\ell'}(\hat{n}\cdot\hat{n}') = \frac{4\pi}{2\ell+1}\delta_{ll'} \tag{8}$$

we obtain

$$C_\ell = \frac{1}{4\pi}\int d^2\hat{n} d^2\hat{n}'\, P_\ell(\hat{n}\cdot\hat{n}')\Delta T(\hat{n})\Delta T(\hat{n}') \tag{9}$$

Notice that C_0 is automatically zero by the definition of $\Delta T(\hat{n})$ i.e., (1) and (2). How about C_1?

In our earlier article on the history of astronomy, we mentioned that the peculiar motion of the Earth relative to the CMB rest frame causes "CMB dipole anisotropy." In our earlier article on spherical harmonics, we explained that $\ell = 1$ corresponds to the dipole moment. So, let's prove the claim that our peculiar motion causes a non-zero C_1.

Recall from the last article that the number of photon is given as follows in the CMB rest frame.

$$dN = 2\frac{d^3p d^3x}{h^3}\frac{1}{\exp(E/kT)-1} \tag{10}$$

where we have $E = |p|c$, and d^3x denotes the volume element. However, as the Earth is moving relative to the CMB rest frame, we observe different energy and momentum of photons. If we denote the Earth coordinate by $'$, we can write

$$dN' = 2\frac{d^3p' d^3x'}{h^3}\frac{1}{\exp(E'/kT')-1} \tag{11}$$

However, we know that the combination d^3pd^3x/h^3 is Lorentz invariant. It gives you the number of available state, so it should not depend on the coordinate. As we have $dN = dN'$. We have

$$\frac{E}{kT} = \frac{E'}{kT'} \tag{12}$$

So, let's find T', the CMB temperature we observe on the Earth. For convenience, let's choose our cooridnate system that the Earth is moving in z-direction with v_z. Then, the Lorentz transformation says

$$E' = \gamma(E - v_z p_z), \qquad E = \gamma(E' + v_z p'_z) \tag{13}$$

Thus, we have

$$T' = \frac{T}{\gamma\left(1 + v_z \frac{p'_z}{E'}\right)} \tag{14}$$

Notice that we expressed T' in terms of p'_z and E' instead of p_z and E because p'_z and E' are the ones that we actually observe here on the Earth.

If we look at the direction $\hat{n}(\theta, \phi)$, the momentum of photon coming from that direction is given by

$$p'_z = -\frac{E'}{c} \cos\theta \tag{15}$$

Thus, we obtain

$$T' = T\sqrt{1-\beta^2}(1 + \beta\cos\theta + \beta^2\cos^2\theta + \cdots) \tag{16}$$

which implies

$$\Delta T = T\left(-\frac{\beta^2}{6} + \beta P_1(\cos\theta) + \frac{2\beta^2}{3}P_2(\cos\theta) + \cdots\right) \tag{17}$$

Problem 1. Check this! Hint: $P_1(\cos\theta) = \cos\theta$, $P_2(\cos\theta) = (3\cos^2\theta - 1)/2$

Thus, we see that the CMB dipole anisotropy is on the order of our peculiar velocity divided by c. From the dipole anisotropy, astronomers obtained $\beta \approx 0.013$, which corresponds to the speed of 370 km/s.

Of course, in the actual analysis, we cannot presuppose that the Earth is moving in the z-direction. In the actual analysis, we calculate a_{10}, a_{1-1} and a_{11} from (4). This gives three independent components, which can be related to the three components (i.e., x, y, and z) of the velocity of the Earth.

Notice also that the velocity of the Earth also contributes to P_2, in other words, to C_2. This contribution is around 10% of actual C_2, which is not quite negligible.