Cauchy's integral formula and Cauchy's residue theorem

Consider a line integral along a closed curve C as follows. We have,

$$\oint_C f(z)dz = \oint (u+iv)(dx+idy) = \oint (udx - vdy) + i\oint (vdx + udy) \tag{1}$$

Such a closed curve on which complex line integral is performed is called "contour." If $f(z)$ is differentiable on the area which the contour C encompasses we can use Green's theorem as follows. We have:

$$\oint_C f(z)dz = \int \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy = 0 \tag{2}$$

where we have used Cauchy-Riemann equations in the last step. Surprisingly, this is exactly zero. This is the power of Cauchy-Riemann equations as advertised!

Now, we consider the case in which the integrand $f(z)$ is not differentiable at some points inside the area which C encompasses. First, we will consider a case in which $f(z)$ is not differentiable at a single point $z=a$ inside the area. Furthermore, we will consider a case in which $f(z)$ is given as follows:

$$f(z) = \frac{g(z)}{z-a} \tag{3}$$

where $g(z)$ is a function that satisfies:

$$\lim_{z\to a} g(z) = g(a), \qquad g(a) \neq 0 \tag{4}$$

Why we consider such a form of integrand will be clear later. Notice also that such form of $f(z)$ satisfies our earlier criteria that it is not differentiable at the point $z=a$. Now,

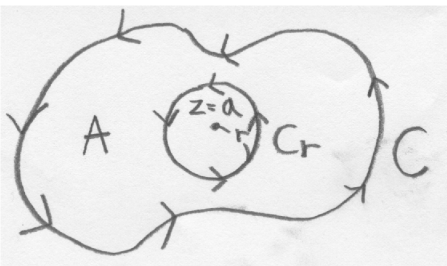

Figure 1: Contour integral

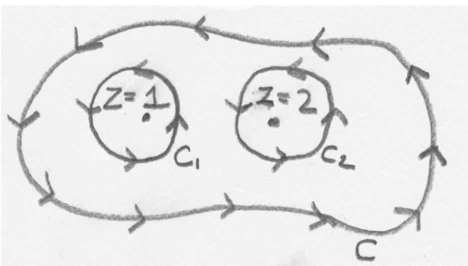

Figure 2: Multiple poles

see Fig.1. You see the closed curve C on which we will perform line integral. $z = a$ is also marked in the figure. Now comes the crucial part. As the $f(z)$ is differentiable in the region A bounded by C_r and C, the area integral in that region will be zero. In other words, by Green's theorem, we have:

$$0 = \oint_C f(z)dz + \oint_{-C_r} f(z)dz = \oint_C f(z)dz - \oint_{C_r} f(z)dz \tag{5}$$

Therefore, we conclude, the line integral on C is the same as the line integral on C_r. So, we can just as well calculate the latter. Here, C_r is a circle with radius r centered at $z = a$. Then, we have $z = re^{i\theta} + a$. Plugging this in, we get:

$$\oint_{C_r} \frac{g(z)}{z-a} dz = \oint_{C_r} \frac{g(re^{i\theta} + a)}{re^{i\theta}} ire^{i\theta} d\theta = i \oint_{C_r} g(re^{i\theta} + a) d\theta \tag{6}$$

Now, notice that we can take r as small as possible. Thus, we obtain:

$$\oint_C \frac{g(z)}{z-a} dz = \lim_{r\to 0} i \oint_{C_r} g(re^{i\theta} + a) d\theta = i \int_0^{2\pi} g(a) d\theta = 2\pi i g(a) \tag{7}$$

In other words,

$$\oint_C \frac{g(z)}{z-a} dz = 2\pi i g(a) \tag{8}$$

This is called "Cauchy's integral formula." C has to be counter-clockwise. If it's clockwise, we get an extra negative sign. Also, such $g(a)$ is called a "residue" around a.

Problem 1. Show that the residues of

$$\oint \frac{dz}{(z-2)(z+3)} \tag{9}$$

around 2 and around -3 are $1/5$ and $-1/5$ respectively.

Also, if a function $f(z)$ is given as follows,

$$f(z) = \frac{h(z)}{(z-b)^n} \tag{10}$$

for some positive integer n, we say b is a pole of f with order n, provided that $h(b) \neq 0$.

Now, we can perform contour integral in the case in which there are multiple poles. For example, see Fig.2. We want to calculate the following:

$$\oint_C \frac{z^2}{(z-1)(z-2)} dz \tag{11}$$

From the same reason as before, this integral is equal to the integral on C_1 and C_2. Therefore, the above equation becomes:

$$\oint_{C_1} \frac{z^2/(z-2)}{z-1} dz + \oint_{C_2} \frac{z^2/(z-1)}{z-2} dz = 2\pi i (1^2/(1-2) + 2^2/(2-1)) = 6\pi i \tag{12}$$

The fact that a contour integral is given by $2\pi i$ times the sum of residues inside the contour is known as "Cauchy's residue theorem." Notice that there is no contribution from the poles outside the contour.

Problem 2. Prove the following by taking the similar step as (6).

$$\oint_C \frac{1}{(z-a)^{k+1}} = 0, \quad \text{for a positive integer k} \tag{13}$$

where the counter-clockwise contour C encircles $z = a$.

Problem 3. Use the above formula and Taylor expansion prove following:

$$\oint_C \frac{h(z)}{(z-a)^{k+1}} dz = 2\pi i \frac{h^{(k)}(a)}{k!} \tag{14}$$

where $h^{(k)}(z)$ denotes kth derivatives of $h(z)$, and the counter-clockwise C encircles $z = a$ but no other poles. In other words, the residue of the above expression around $z = a$ is given by $h^{(k)}(a)$.

Problem 4. Calculate the residues around $z = 2$ and $z = 3$ for the following integration.

$$\oint \frac{dz}{(z-2)^2(z-3)} \tag{15}$$

Summary

- For a holomorphic function $g(z)$, Cauchy's integral formula says

  $$\oint_C \frac{g(z)}{z-a} dz = 2\pi i g(a)$$

  where C is a counter-clockwise contour that encircles $z = a$. Here, $g(a)$ is called a "residue."

- If $f(z)$ is given as follows,

  $$f(z) = \frac{h(z)}{(z-b)^n}$$

  for some positive integer n, we say b is a pole of f with order n, provided that $h(b) \neq 0$.

- Cauchy's residue theorem says that a contour integral is given by $2\pi i$ times the sum of residues inside the contour.

- For a positive integer k, we have

  $$\oint_C \frac{1}{(z-a)^{k+1}} = 0$$