Coriolis force, revisiting

In this article, we will approach Coriolis force quantitatively by closely following *Analytical Mechanics* by Fowles and Cassiday.

Remember that Coriolis force is an inertial force, a force that doesn't actually exist, but you feel as you are accelerating (rotating in case of Coriolis force). In other words, the coordinate system is rotating.

Suppose you have a position vector $\vec{r}$. In an inertial frame, it will seem as

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{1}$$

On the other hand, the same vector $\vec{r'} = \vec{r}$ will seem in the rotating frame as

$$\vec{r'} = x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}' \tag{2}$$

where the primes denote the rotating frame.

Now, let's see how the time derivative of $\vec{r}$ and $\vec{r'}$ look like. From (1) and (2) we have:

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k} = \frac{d\vec{r'}}{dt} = \left(\frac{dx'}{dt}\hat{\imath}' + \frac{dy'}{dt}\hat{\jmath}' + \frac{dz'}{dt}\hat{k}'\right) + \left(x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt}\right) \tag{3}$$

Notice that for the unprimed coordinate the unit vectors are not changing so their time derivatives were absent in the above equation, while for the primed coordinate the unit vectors are changing, so their time derivatives must be explicitly present in the above equation.

Given this, let's denote the terms in the first parenthesis in the above equation by

$$\left(\frac{d\vec{r'}}{dt}\right)_{rot} = \frac{dx'}{dt}\hat{\imath}' + \frac{dy'}{dt}\hat{\jmath}' + \frac{dz'}{dt}\hat{k}' \tag{4}$$

Notice that this is the actual time derivative of $\vec{r'}$ measured by the rotating coordinate.

What still remains to be done is obtaining an expression for the second parenthesis of (3). This means that we need to calculate $d\hat{i}'/dt$, $d\hat{j}'/dt$, $d\hat{k}'/dt$. See Fig.1. Here, we calculate $d\hat{i}'/dt$. The vector $\vec{\omega}$ denotes the axis of rotation. ϕ is the angle between $\vec{\omega}$ and $\hat{\imath}'$. After rotating i' by infinitesimal $\Delta\theta$, $\hat{\imath}'$ changes by $\Delta\hat{\imath}'$. From the figure, it is obvious that

$$|\Delta\hat{\imath}'| = (|\hat{\imath}'|\sin\phi)\Delta\theta \tag{5}$$

Dividing by Δt, we get:

$$|\frac{d\hat{i}'}{dt}| = \sin\phi\frac{d\theta}{dt} = (\sin\phi)\omega \tag{6}$$

Notice also that the direction of $\Delta\hat{\imath}'$ is perpendicular to both $\vec{\omega}$ and $\hat{\imath}'$. Along with the above equation which gives the magnitude of $d\hat{\imath}'/dt$, this implies, from the definition of the cross product,

$$\frac{d\hat{\imath}'}{dt} = \vec{\omega} \times \hat{\imath}' \tag{7}$$

And, similarly for $d\hat{\jmath}'/dt$ and $d\hat{k}'/dt$. Using this, the second parenthesis of (3) can be re-expressed as

$$x'(\vec{\omega} \times \hat{\imath}') + y'(\vec{\omega} \times \hat{\jmath}') + z'(\vec{\omega} \times \hat{k}') = \vec{\omega} \times (x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}') = \vec{\omega} \times \vec{r}' \tag{8}$$

Plugging (4) and (8) into (3), we have:

$$\frac{d\vec{r}}{dt} = \left(\frac{d\vec{r}'}{dt}\right)_{rot} + \vec{\omega} \times \vec{r}' \tag{9}$$

By taking the same step as deriving the above equation, we can derive the following universal formula

$$\frac{d\vec{Q}}{dt} = \left(\frac{d\vec{Q}}{dt}\right)_{rot} + \vec{\omega} \times \vec{Q} \tag{10}$$

where $\vec{Q}$ is any vector.

Now, we are ready to derive the expression for the Coriolis force. Let $\vec{v} = d\vec{r}/dt$ and $\vec{v}' = d\vec{r}'/dt$. Then, (9) can be re-expressed as

$$\vec{v} = \vec{v}' + \vec{\omega} \times \vec{r}' \tag{11}$$

Furthermore, by plugging $\vec{v}$ into $\vec{Q}$ for (10), we get:

$$\frac{d\vec{v}}{dt} = \left(\frac{d\vec{v}}{dt}\right)_{rot} + \vec{\omega} \times \vec{v} \tag{12}$$

Plugging (11) to the above equation, we get:

$$\begin{aligned}
\frac{d\vec{v}}{dt} &= \left(\frac{d}{dt}\right)_{rot} (\vec{v}' + \vec{\omega} \times \vec{r}') + \vec{\omega} \times (\vec{v}' + \vec{\omega} \times \vec{r}') \\
&= \left(\frac{d\vec{v}'}{dt}\right)_{rot} + \left(\frac{d(\vec{\omega} \times \vec{r}')}{dt}\right)_{rot} + \vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}') \\
&= \left(\frac{d\vec{v}'}{dt}\right)_{rot} + \left(\frac{d\vec{\omega}}{dt}\right)_{rot} \times \vec{r}' + \vec{\omega} \times \left(\frac{d\vec{r}'}{dt}\right)_{rot} + \vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}')
\end{aligned} \tag{13}$$

Remembering $\vec{v}' = d\vec{r}'/dt$ and using the notation $\vec{a} = d\vec{v}/dt$ and $\vec{a}' = (d\vec{v}'/dt)_{rot}$ which are reasonable as the former is the real acceleration and the latter is the apparent acceleration felt in rotating frame, the above equation can be re-written as:

$$\vec{a} = \vec{a}' + \left(\frac{d\vec{\omega}}{dt}\right)_{rot} \times \vec{r}' + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}' \tag{14}$$

If you plug $\vec{\omega}$ into (10), you get $d\vec{\omega}/dt = (d\vec{\omega}/dt)_{rot}$. (**Problem 1.** Check this!) Thus, we obtain:

$$\vec{a} = \vec{a}' + \frac{d\vec{\omega}}{dt} \times \vec{r}' + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}') \tag{15}$$

Therefore, the acceleration the observer in rotating frame feels is given as follows:

$$\vec{a}' = \vec{a} - \frac{d\vec{\omega}}{dt} \times \vec{r}' - 2\vec{\omega} \times \vec{v}' - \vec{\omega} \times (\vec{\omega} \times \vec{r}') \tag{16}$$

The first term is due to the real acceleration, the second term is due to the change of rotation speed, the third term is called Coriolis force and the fourth term is called centrifugal force.

Problem 2. From the above equation, convince yourself that the direction of the Coriolis force in the Northern Hemisphere is rightward with respect to the moving direction. Also, convince yourself that the centripetal force given in the above equation acts on the object in the correct direction (away from the rotation axis.)

Summary

- Coriolis force is proportional to the rotation angular velocity and the moving velocity.