Dirac string

As advertised, in this article we will prove that the existence of magnetic monopole implies that the electric charges are quantized. To this end, we need to find the vector potential of magnetic field in the presence of magnetic monopole.

However, notice that such a vector potential doesn't exist, since the presence of magnetic monopole necessarily implies that the divergence of magnetic field doesn't vanish. In other words, as the vector potential $\vec{A}$ is defined by $\vec{B} = \nabla \times \vec{A}$ and the divergence of the curl of any vector is always zero, only the divergenceless magnetic field can be expressed in terms of a vector potential.

Nevertheless, we should not give up. We must do as far as we can. Therefore, we will find a vector potential that gives the correct magnetic field except for an infinitesimal region called "Dirac string." See Fig.1. The magnetic monopole is located at the origin O and Dirac string is located along negative z-axis. We also have a Gaussian surface which is given by a sphere surrounding the magnetic monopole subtracted by an infinitesimal hole that is located at the cross section of Dirac string and the sphere. We will call this Gaussian surface S. Then, naturally the boundary of S is given by the infinitesimal hole.

Now, let's do some exercise. First, we will say that the magnetic charge of the magnetic monopole concerned is given by g. Then, we will have:

$$g = \int_S \vec{B} \cdot d\vec{A} \tag{1}$$

Of course, this formula is possible because the small hole is infinitesimal so that no magnetic field leaks through it. Let's proceed, we have:

$$g = \int_S (\nabla \times \vec{A}) \cdot d\vec{A} = \int_{\partial S} \vec{A} \cdot d\vec{s} \tag{2}$$

where ∂S denotes the boundary of S, which is the infinitesimal hole.

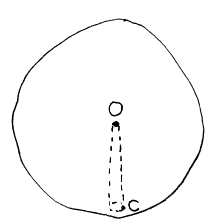

Figure 1: Magnetic monopole with Gaussian surface and Dirac string

Given this, recall what we have derived in "The Aharonov-Bohm effect." In the presence of magnetic field, the wave function ψ is given by

$$\psi(\vec{r}) = e^{if}\psi_0(\vec{r}) \tag{3}$$

where

$$f = \frac{q}{\hbar}\int_{\vec{r}_i}^{\vec{r}} \vec{A}(\vec{r}') \cdot d\vec{r}' \tag{4}$$

and ψ_0 is the wave function in the absence of magnetic field. However, there is a subtle difference between the former case and the present case. In the former case, the line integration of vector potential was along the path. In the presence case, it is along *any* path; in solving Schrödinger equation, we are now taking spatial derivatives along any path, not just the path in which the particle is actually moving.

Given this, let's say an object with charge q is sitting at C. If we plug $\vec{r}_i = \vec{r} = C$ and (2) into (4), we get:

$$f = \frac{qg}{\hbar} \tag{5}$$

In other words, if we integrate along the hole one time, coming back to the original position, we get this much of extra phase.

However, the wave function must be single-valued; the wave function at a certain point cannot take two numbers as its value. Therefore, the extra phase must be $2n\pi$ where n is an integer as

$$e^{if} = e^{2n\pi i} = 1 \tag{6}$$

Therefore, from $f = qg/\hbar = 2\pi n$, we conclude:

$$\frac{qg}{2\pi\hbar} = n \tag{7}$$

This is called "Dirac quantization condition." In other words, we have $q = (2\pi n\hbar)/g$. We see here that q, the electric charge, is an integer multiple of unit charge $(2\pi\hbar)/g$. In other words, if there is a single magnetic monopole in the universe, the electric charge is quantized, being integer multiples of a unit charge. This is exactly what we see in the nature! The electric charge is quantized in such a manner. Of course, this doesn't necessarily prove the presence of the magnetic monopole, as there could be other origins of quantization of electric charge.

Let us conclude this article with some comments. In this article, we could not find a vector potential that correctly describes magnetic field in all regions in the presence of magnetic monopole. This is not surprising, if you correctly solved Problem 3 in "Electromagnetic duality." Nevertheless, we can find such a vector potential that describes so "locally." In such a case, we say we cannot find a global vector potential but find a local vector potential. After all, in our case, if we additionally find just one vector potential whose Dirac string lies on positive z-axis, it can describe the magnetic field at the negative z-axis, the region which our vector potential could not describe the magnetic field. In other words, these two local vector potentials are enough to describe magnetic monopole's magnetic field in the whole region. Mathematically, Dirac monopole is well described by what is called "fiber bundle." If you are interested, you may want to read *Geometry, Topology and Physics* by Nakahara.

Also, as we mentioned in "Maxwell's equation in differential forms," we have electric and magnetic fields in string theory as well. They also follow Dirac quantization condition.

Problem 1. If a magnetic monopole exists, would its magnetic charge also be quantized in such a manner as electric charges are quantized? Why or why not?

Finally, let us explain to you why electromagnetic duality is S-duality. As explained in "Electron magnetic moment" we use Taylor-series in α, the fine structure constant, to calculate a lot of quantities in quantum electrodynamics. In such a case, we say that α plays the role of "coupling constant." What would be the magnetic analog of α? Remember that α is proportional to the Coulomb interaction between two electric charges e. Thus, roughly speaking, if there is the magnetic version of fine structure constant, it should be proportional to g^2. However, remember that $g^2 = \frac{4\pi^2\hbar^2}{e^2}$. In other words, the magnetic coupling constant is inversely proportional to the electric coupling constant. This is exactly S-duality. Notice also that as the electric interactions get stronger, the magnetic interactions get weaker and vice versa. For this reason, S-duality is also called "strong-weak duality."

Summary

- Dirac showed that, if there exists a magnetic monopole with magnetic charge g in our universe, then ge must be integer multiple of a fundamental unit.