Ehrenfest theorem

1 Schrödinger picture and Heisenberg picture

There are two equivalent pictures in quantum mechanics: Schrödinger picture and Heisenberg picture

In Schrödinger picture, the operator doesn't evolve, while the state vector evolves. In particular, we have seen that the state vector evolves the following way:

$$\psi_S(t) = e^{-iHt/\hbar}\psi_S(t=0) \tag{1}$$

where the subscript S denotes "Schrödinger picture." Similarly, the fact that the operator doesn't evolve can be expressed as:

$$A_S(t) = A_S(t=0) \tag{2}$$

Let's calculate how the expectation value evolves:

$$\begin{gathered}\langle A(t)\rangle = \langle\psi_S(t)|A_S(t)|\psi_S(t)\rangle = \langle\psi_S(0)|(e^{-iHt/\hbar})^\dagger A_S(0)e^{-iHt/\hbar}|\psi_S(0)\rangle \\ = \langle\psi_S(0)|e^{iH^\dagger t/\hbar}A_S(0)e^{-iHt/\hbar}|\psi_S(0)\rangle = \langle\psi_S(0)|e^{iHt/\hbar}A_S(0)e^{-iHt/\hbar}|\psi_S(0)\rangle\end{gathered} \tag{3}$$

On the other hand, in Heisenberg picture, it is the operator that evolves, not the state vector. State vector doesn't evolve in Heisenberg picture. (i.e. $|\psi_H(t)\rangle = |\psi_H(0)\rangle$) Then, we ask: "How does the operator evolve?" The answer is given in our previous article:

$$\frac{dA_H}{dt} = \{A_H, H\} = \frac{[A_H, H]}{i\hbar} \tag{4}$$

where the subscript H denotes "Heisenberg picture." Now, we will solve the above differential equation. The solution is given by following:

$$A_H(t) = e^{iHt/\hbar}A_H(0)e^{-iHt/\hbar} \tag{5}$$

Let's verify this:

$$\begin{aligned}\frac{d}{dt}(e^{-iHt/\hbar}A_H(0)e^{iHt/\hbar}) &= -\frac{iH}{\hbar}e^{-iHt/\hbar}A_H(0)e^{iHt/\hbar} - e^{-iHt/\hbar}A_H(0)e^{iHt/\hbar}\frac{iH}{\hbar} \\ &= \frac{[A_H, H]}{i\hbar}\end{aligned} \tag{6}$$

Then, the expectation value in Heisenberg picture is given by:

$$\langle A(t)\rangle = \langle\psi_H(t)|A(t)|\psi_H(t)\rangle = \langle\psi_H(0)|e^{iHt/\hbar}A_H(0)e^{-iHt/\hbar}|\psi_H(0)\rangle \quad (7)$$

Comparing with (3), we see that two pictures are equivalent.

2 Ehrenfest theorem

Ehrenfest theorem states that the expectation value of operator satisfies classical equation of motion. In other words,

$$\frac{d\langle A\rangle}{dt} = \langle\{A, H\}\rangle \quad (8)$$

Using, the relation between the Poisson bracket and the commutator, this is equal to:

$$\frac{d\langle A\rangle}{dt} = \frac{\langle[A, H]\rangle}{i\hbar} \quad (9)$$

Let's prove this. First, the easy case. Heisenberg picture.

$$\frac{d}{dt}\langle\psi_H(t)|A_H(t)|\psi_H(t)\rangle$$
$$= \langle\psi_H(t)|\frac{[A_H, H]}{i\hbar}|\psi_H(t)\rangle = \frac{\langle[A, H]\rangle}{i\hbar} \quad (10)$$

where we used (4) from the first line to the second line, and the fact that the state vector in Heisenberg picture doesn't evolve. So, this completes the proof.

In case of the Schrödinger picture, we have:

$$\frac{d}{dt}\langle\psi_S(t)|A_S(t)|\psi_S(t)\rangle \quad (11)$$
$$= \langle\psi_S(t)|(iH/\hbar)A_S|\psi_S(t)\rangle + \langle\psi_S(t)|A_S(-iH/\hbar)|\psi_H(t)\rangle \quad (12)$$
$$= \frac{\langle[A, H]\rangle}{i\hbar} \quad (13)$$

where we used the followings from the first line to the second line:

$$\frac{\partial}{\partial t}|\psi_S\rangle = -\frac{iH}{\hbar}|\psi_S\rangle \quad (14)$$
$$\frac{\partial}{\partial t}\langle\psi_S| = \langle\psi_S|(\frac{-iH}{\hbar})^\dagger = \langle\psi_S|\frac{iH}{\hbar} \quad (15)$$

In conclusion, Ehrenfest theorem clearly shows that quantum mechanics reduces to classical mechanics in macroscopic world.

Summary

- In Schrödinger picture, the operator doesn't evolve, while the state vector evolves.
- In Heisenberg picture, it is the operator that evolves, not the state vector.
- Ehrenfest theorem states that the expectation value of operator satisfies classical equation of motion.