Euler's formula and hyperbolic functions

At first glance, it doesn't make sense that an exponent can be an imaginary number, since multiplying something an imaginary number times doesn't really make sense. Nevertheless, we can use Taylor series to calculate one. So, let's calculate $e^{i\theta}$ using Taylor expansion.

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \end{aligned}$$

Therefore, we conclude:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1}$$

This is Euler's formula.

Problem 1. Show the following.

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{2}$$

Problem 2. Thereby, obtain the values for $\cos i$ and $\sin i$.

Problem 3. Using Euler's formula, show that $\ln(-1) = i\pi$. A careful reader may notice that $\ln(-1)$ can be other values as well such as $-i\pi$ or $3i\pi$. Nevertheless, we can single out one of them, when it is necessary, in a similar manner as we can give unique values to inverse trigonometric functions, even though trigonometric functions can give the same value for different inputs.

Problem 4. Hyperbolic cosine function (pronounced as "cosh"), hyperbolic sine function (pronounced as "sinch"), and hyperbolic tangent function (pronounced as "tanch") are defined as follows:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}, \qquad \tanh x = \frac{\sinh x}{\cosh x} \tag{3}$$

Given these, show the followings:

$$\cosh ix = \cos x, \qquad \sinh ix = i\sin x, \qquad \cosh^2 x - \sinh^2 x = 1 \tag{4}$$

$$\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y \tag{5}$$

$$\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y \tag{6}$$

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x\tanh y} \tag{7}$$

Problem 5. Use (2) to show the following.

$$2\cos^2\theta = 1+\cos 2\theta, \qquad 4\cos^3\theta = 3\cos\theta + \cos 3\theta \tag{8}$$

Problem 6. We will now simplify the following expression.

$$S = \cos\theta + \cos 2\theta + \cos 3\theta + \cdots + \cos n\theta \tag{9}$$

This is equal to

$$S = \mathrm{Re}\left(e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \cdots + e^{in\theta}\right) \tag{10}$$

$$= \mathrm{Re}\left(e^{i\theta}\frac{1-e^{in\theta}}{1-e^{i\theta}}\right) = \mathrm{Re}\left(e^{i\theta}\frac{e^{in\theta/2}}{e^{i\theta/2}}\frac{(e^{in\theta/2}-e^{-in\theta/2})/2i}{(e^{i\theta/2}-e^{-i\theta/2})/2i}\frac{e^{in\theta/2}}{e^{i\theta/2}}\right) \tag{11}$$

$$= \mathrm{Re}\left(e^{i(n+1)\theta/2}\frac{\sin(n\theta/2)}{\sin(\theta/2)}\right) = \cos[(n+1)\theta/2]\frac{\sin(n\theta/2)}{\sin(\theta/2)} \tag{12}$$

where, from (10) to (11) we used the summation formula for geometric series. Similarly, simplify the following expression.

$$T = \sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta \tag{13}$$

Problem 7. Show that the solution to $x^n = 1$ for a positive integer n is given by

$$x = e^{2\pi ik/n} \tag{14}$$

where $k = 0, 1, 2, \cdots n-1$.

Problem 8. Find the solution to $x^4 = -1$. (Hint[1])

Summary

- $e^{ix} = \cos x + i\sin x$ which implies $\cos x = \dfrac{e^{ix}+e^{ix}}{2}, \qquad \sin x = \dfrac{e^{ix}-e^{-ix}}{2}$.
- $\cosh x = \dfrac{e^x + e^{-x}}{2}, \qquad \sinh x = \dfrac{e^x - e^{-x}}{2}, \qquad \tanh x = \dfrac{\sinh x}{\cosh x}$
- $\cosh^2 x - \sinh^2 x = 1$.

[1] Use the fact that $-1 = e^{i\pi} = e^{3i\pi} = e^{5i\pi} = e^{7i\pi}$.