Faraday's law of induction and the Lorentz force

In our earlier article "Faraday's law of induction from the point of view of magnetic force on moving charge," we interpreted as if Faraday's law emerges from the magnetic force when combined with the old Galilean principle (i.e. we do not know which one is moving). In this article, we will quantitatively prove this.

First, consider the case in which the magnet is not moving, but the circuit is moving. Then, the rate of change of flux through a circuit is given as follows

$$\frac{d\Phi_B}{dt} = \frac{d\int_\Sigma \vec{B}\cdot d\vec{A}}{dt} = \int_\Sigma \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A} + \int_{d\Sigma/dt} \vec{B}\cdot d\vec{A} = \int_{d\Sigma/dt} \vec{B}\cdot d\vec{A} \tag{1}$$

where we used the fact $\partial\vec{B}/\partial t = 0$ as the magnetic field doesn't change over time as magnet is not moving.

The above expression is equal to

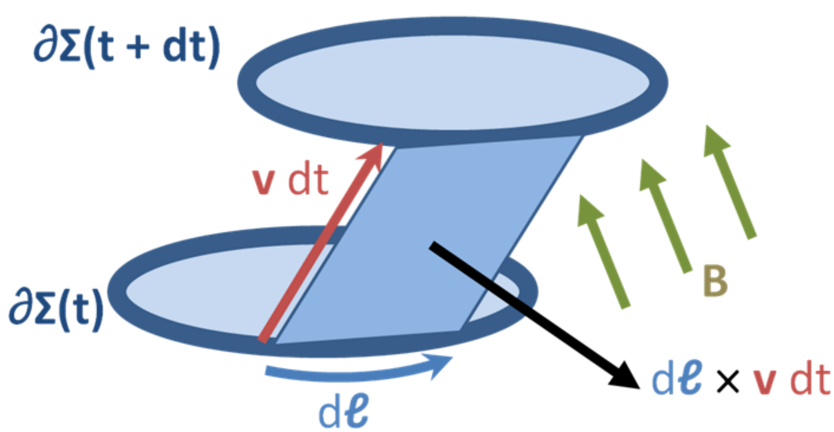
$$\frac{\int_{\Sigma(t+dt)-\Sigma(t)} \vec{B}\cdot d\vec{A}}{dt} \tag{2}$$

Let's calculate the numerator. See the figure. As there is no source for magnetic field, we have

$$\int_{\Sigma(t)+\text{side}-\Sigma(t+dt)} \vec{B}\cdot d\vec{A} = 0 \tag{3}$$

where side denotes "the side" of "the cylinder." Thus, (2) becomes

$$\frac{\int_{\text{side}} \vec{B}\cdot d\vec{A}}{dt} \tag{4}$$



From the figure, it is obvious that the area element is given by

$$d\vec{A} = d\vec{l} \times \vec{v}dt \tag{5}$$

where $\vec{v}$ is the velocity of the circuit. Therefore, the flux is given by

$$\vec{B} \cdot (d\vec{l} \times \vec{v}dt) = -dt\, d\vec{l} \cdot (\vec{v} \times \vec{B}) = (\vec{v} \times \vec{B}) \cdot dt\, d\vec{l} \tag{6}$$

Summarizing, we obtain

$$-\frac{d\Phi_B}{dt} = \int_{\partial\Sigma} (\vec{v} \times \vec{B}) \cdot d\vec{l} \tag{7}$$

Given this, consider a charge q on the circuit. The charge moves because circuit moves with velocity $\vec{v}$. Thus, the charge will experience the Lorentz force $\vec{F} = q\vec{v} \times \vec{B}$. If the charge rotates once around the circuit due to this magnetic force, it will obtain the following energy.

$$\int_{\partial\Sigma} \vec{F} \cdot d\vec{l} = q \int_{\partial\Sigma} (\vec{v} \times \vec{B}) \cdot d\vec{l} \tag{8}$$

Using the fact that the electric energy gain upon one rotation divided by q is the electromotive force $\mathcal{E}$, we get:

$$\mathcal{E} = \int_{\partial\Sigma} (\vec{v} \times \vec{B}) \cdot d\vec{l} \tag{9}$$

Comparing with (7), we conclude

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \tag{10}$$

which is exactly Faraday's law. We derived this when the magnet is not moving. Now, consider another observer who is moving relatively to the magnet. He will see that the magnet is moving, and he will also see that circuit is moving with a different velocity than $\vec{v}$. Nevertheless, he will always see the same value for Φ_B with the one who sees that the magnet is not moving. Therefore, the former will agree with the latter, as far as $d\Phi_B/dt$ is concerned.[1] Thus, the former will also see that (10) is valid.

(The figure is from `https://commons.wikimedia.org/wiki/File:Faraday_Area.PNG`)

Summary

- By considering the Lorentz force that charges inside a wire feel upon moving relatively to a stationary magnetic field, we can derive Faraday's law that we have an induced electromotive force if the magnetic flux through the wire changes.

[1] Of course, non-relativistically. We will not consider relativistic case for simplicity.