What is Fourier series?

In our earlier article "Polar coordinate and trigonometric functions defined over all values," we briefly mentioned what Fourier series was. In this article, we will give you an explicit example.

To this end, let us show you how the graph of sine and cosine functions look like. Fig.1 is the graph of sine function. Fig.2 is the graph of cosine function. As explained earlier, they are periodic functions with period 2π. (Remember that the period of sine and cosine functions was given by 360 degree. If we use radian then this becomes 2π.)

Would there be any way to accommodate a different period in the sine function? The answer is yes. See Fig.3. This is the graph of $\sin 2x$. The period was halved, and became π. Let's check this:

$$\sin(2x) = \sin(2x + 2\pi) = \sin(2(x + \pi)) \tag{1}$$

Now, let's consider a periodic function in Fig.4. We will demonstrate that it is possible to express this function as sums of sine or cosine function. To this end, let's find out what the period of this function is. If we call this function $f(x)$, it is easy to see that it is 2 since it satisfies $f(x) = f(x + 2)$. What form of sine and cosine function satisfies this property? Let's express an arbitrary sine function as $\sin kx$, with k to be determined soon. Then, we have:

$$\sin kx = \sin k(x + 2) = \sin(kx + 2k) = \sin(kx + 2n\pi) \tag{2}$$

for any integer n. Therefore, we conclude $k = n\pi$. Same can be said about an arbitrary cosine function. Now, let's express the function in Fig. 4 in terms of sum of sine and cosine functions. It should be the following form:

$$f(x) = b_0 + \sum_{n=1}^{\infty} (a_n \sin(n\pi x) + b_n \cos(n\pi x)) \tag{3}$$

We see that this is an infinite sum. Actually, it turns out that

$$a_n = -\frac{1}{n}, \quad b_n = 0, \quad b_0 = 0 \tag{4}$$

The calculation of these coefficients is out of scope for this article. If you are interested, please read the book "Who is Fourier?" or our article "Fourier transformations."

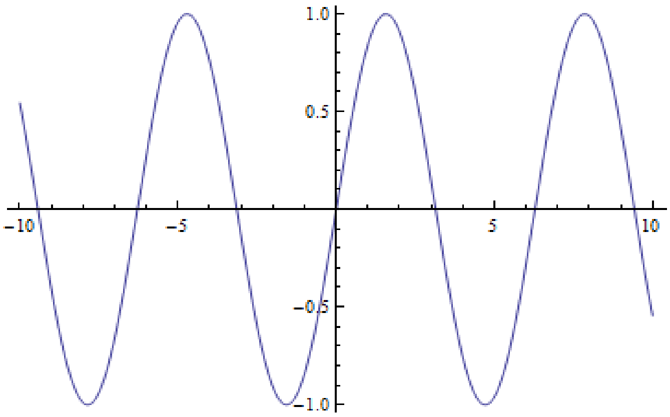

Fig. 1 sin x

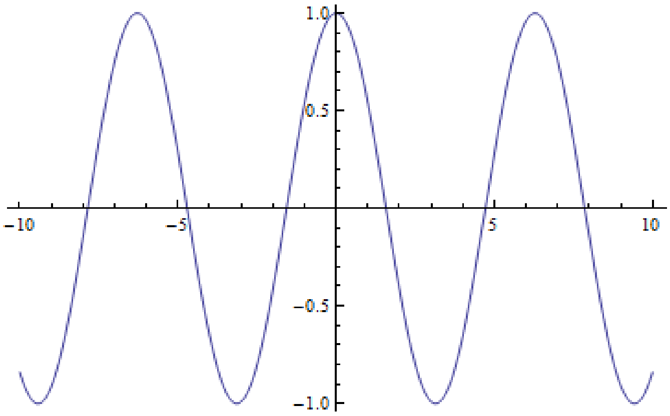

Fig. 2 cos x

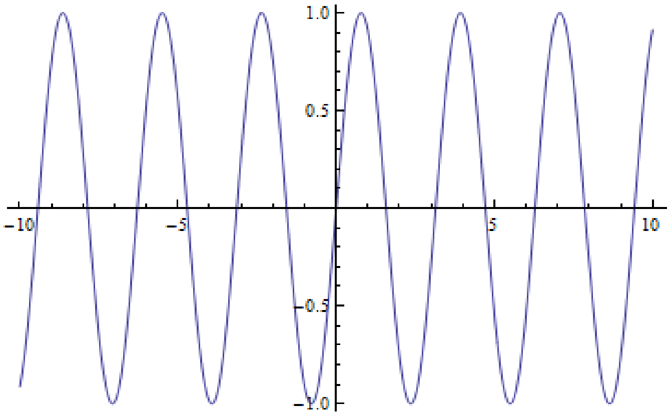

Fig. 3 sin 2x

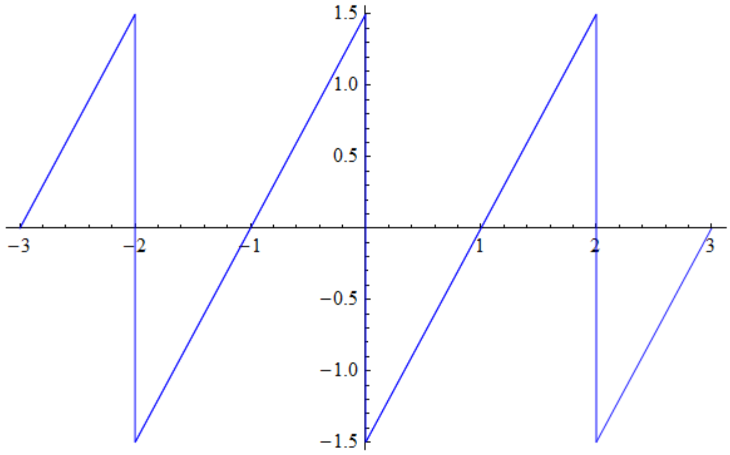

Fig. 4

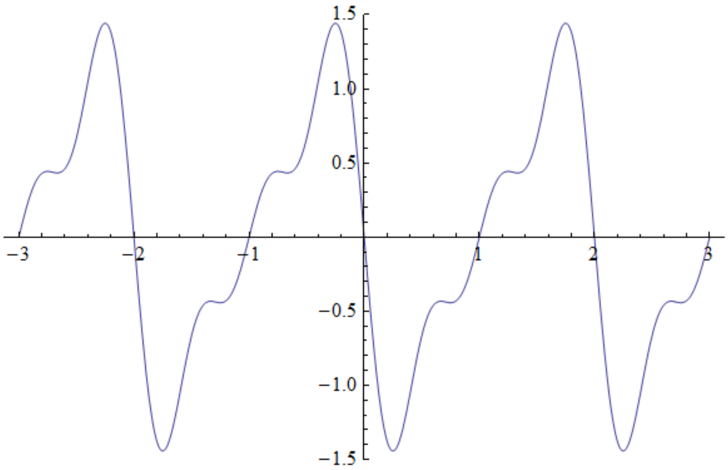

Fig. 5

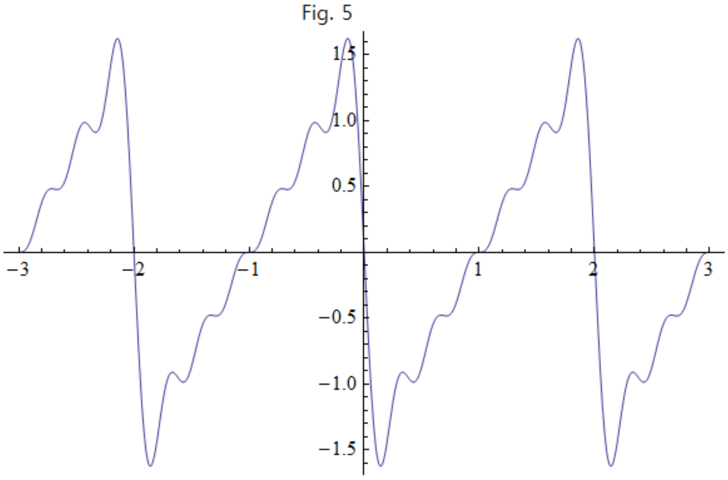

Fig. 6

Now, Fig. 5 is the graph of (3) added up to the first three terms. It doesn't look that similar to Fig. 4. But if you add more term, it will look more similar. Fig. 6 is the graph of (3) added up to the first six terms. Now, it looks more similar. If we add more and more terms it will converge to Fig. 4. This is Fourier series, expressing a periodic function in terms of an infinite sum of sine and cosine functions.

The French mathematician and physicist Jean-Baptiste Joseph Fourier has invented this method for his research in heat transfer and vibrations in the early 19th century. The heat equation which tells you how temperature at each point in a region evolves as time passes admits an easy solution if the temperature distribution in space is in the form of sine and cosine functions. Fourier's ingenuity was realizing that the temperature distribution can be always expressed as a sum of sine and cosine functions, and the solution to such a heat equation is the sum of each solution when the heat distribution is a sine function or cosine function. The analysis of the way general functions can be represented as Fourier series (i.e. a sum of sine and cosine functions) is called "Fourier analysis" or "Fourier transformation." I say, general functions, not just periodic functions, because general functions can be regarded as periodic functions with infinite periods. We will talk more about it in our article "Fourier transformations." Anyhow, Fourier analysis is now very widely used in physics and engineering. If you often listen to radio you may have already seen an example of its everyday applications. Sounds are waves and can be decomposed into their original components (i.e., sine functions and cosine functions) using Fourier transformation. The higher pitch a sound has the higher its frequency. (We will talk more about this in our later article "Light as waves.") See Fig. 7 for a screen of sound analyzer you can see when you listen to radio. The x-axis is the frequency of the sound, and each blue bar represents the Fourier coefficients corresponding to each frequency. For example, if you listen to higher pitched music the blue bars in higher frequency range have higher height. The sound analyzer instantly analyzes the sound by a method called "FFT" (Fast Fourier Transformation) and shows its Fourier coefficients on the screen. Fourier analysis is also used in jpg file, a type of file for picture. Let me explain the basic idea behind it. The image of a digital picture may be stored by pixels. We can assign each pixel a color which, in turn, can be labeled by a number. However, this is not the most efficient way to store the image. Let's say you have an apple and a banana in a picture. Then, the pixels located at the image of apple will be mostly reddish and the pixels located at the image of banana will be mostly yellowish. Therefore, if you stored the image by collecting the color at each pixel, it would be a huge waste of data, because you would have to store repeatedly similar colors repeatedly at close regions. Instead, we can Fourier transform the image and save its coefficients. The coefficients for high frequency (i.e., high n in (3)) will be relatively small as in most cases the color doesn't usually change drastically from one pixel to the next pixel. For

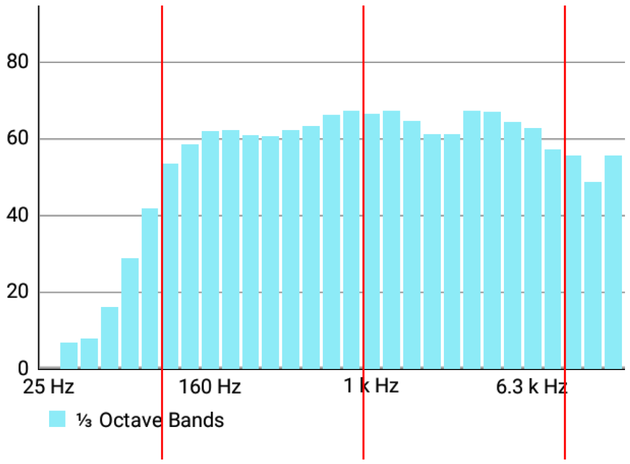

Fig. 7. Sound analyzer

example, in Fig. 6 you see that the graph looks very close to the original one with only six Fourier coefficients. It already gives you a good rough picture of the graph, while giving the coordinates for each point in the graph will take huger data. That way we can take the full advantage of the fact that a pixel at a given location is most likely to have a similar color to the ones at the neighboring pixels. In actual process, there are some more technicalities and steps to improve the storing method, but this is the basic idea behind how Fourier analysis is used in jpg files.

Problem 1. Let $g(x)$ be defined by

$$g(x) = C\sin(x+\theta) \tag{5}$$

Given this, find its Fourier coefficients i.e., a_n, b_n for

$$g(x) = b_0 + \sum_{n=1}^{\infty}(a_n\sin(nx) + b_n\cos(nx)) \tag{6}$$

You need to use the result of the last article.

If you correctly solve this problem you will be additionally able to check the following interesting relation for the Fourier coefficients:

$$a_1^2 + b_1^2 = C^2 \tag{7}$$

This relation is satisfied no matter what θ you choose.

Summary

- A periodic function is expressible in terms of an infinite sum of sine and cosine functions. This series is called Fourier series.

Fig. 7 is from `https://www.flickr.com/photos/dominicspics/27777318489`.