# Fourier transformations

A Fourier series is a way to represent a periodic function by an infinite sum of sine and cosine functions. For example, let us consider a periodic function $f(t)$ for which $f(t) = f(t+T)$. Naturally, the period of this function is $T$. We can write such a periodic function as an infinite sum of sine and cosine functions as follows.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{2\pi}{T} nt\right) + b_n \sin\left(\frac{2\pi}{T} nt\right) \right] \tag{1}$$

You can easily check that this function satisfies the condition $f(t) = f(t+T)$. Now, for simplicity, let us consider the case when $T = 2\pi$. The generalization to an arbitrary $T$ is straightforward. In this case, $f(t)$ can be expressed as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(nt) + b_n \sin(nt) \right] \tag{2}$$

(You will see why it is convenient to include the factor $\frac{1}{2}$ in front of $a_0$ shortly.) Now, let's multiply both sides by $\cos(mt)$ and integrate from 0 to $2\pi$, where $m$ is a non-negative integer. Then we get

$$\begin{aligned} \int_0^{2\pi} f(t)\cos(mt)dt &= \frac{a_0}{2}\int_0^{2\pi} \cos(mt)dt \\ &+ \sum_{n=1}^{\infty} \left[ a_n \int \cos(nt)\cos(mt)dt + b_n \int \sin(nt)\cos(mt)dt \right] \end{aligned}$$

To perform these integrations, the following formulas are useful.

$$\cos(nt)\cos(mt) = \frac{1}{2}\left[\cos(n+m)t + \cos(n-m)t\right] \tag{3}$$

$$\sin(nt)\cos(mt) = \frac{1}{2}\left[\sin(n+m)t + \sin(n-m)t\right] \tag{4}$$

We also need to use the following: (**Problem 1.** Check this!)

$$\int_0^{2\pi} \sin(kt)dt = \int_0^{2\pi} \cos(kt)dt = 0 \quad \text{for a nonzero integer } k \tag{5}$$

$$\int_0^{2\pi} \sin(kt)dt = 0, \quad \int_0^{2\pi} \cos(kt)dt = 2\pi \quad \text{for } k = 0 \tag{6}$$

**Problem 2.** Show (7), (8), and (9)! ($m, n$ both positive integers)

$$\int_0^{2\pi} \cos(nt)\cos(mt)dt = 0 \text{ (if } m \neq n), \quad \pi \text{ (if } m = n) \tag{7}$$

$$\int_0^{2\pi} \sin(nt)\cos(mt)dt = 0 \tag{8}$$

Thus, the only term that survives upon integration is the following

$$\int_0^{2\pi} f(t)\cos(mt)dx = \pi a_m \tag{9}$$

for both when $m = 0$ and when $m > 0$. (You have to check these two cases separately.) Let's re-write this expression slightly differently. We have

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(mt)dt \tag{10}$$

Similarly, by multiplying both sides of (2) by $\sin(mt)$ and following similar steps, we get

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(mt)dt \tag{11}$$

In other words, we have found a way to obtain the coefficients in the Fourier series. Or, we could express the above formulas slightly differently (**Problem 3.** From (10) and (11), show the following! Hint[1])

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)dt \tag{12}$$

When we derive all this again in "Revisiting Fourier transformation," you will get a better perspective on the Fourier transformation.

Anyhow, in the general case (1) with arbitrary $T$, (12) becomes

$$a_m = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi}{T}mt\right)dt \tag{13}$$

$$b_m = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi}{T}mt\right)dt \tag{14}$$

You can easily check that the above formulas make sense without doing the same calculation all again by thinking along the following way. It is natural that you need to integrate $f(t)$ over one cycle. That's how the integration range is determined. From (1), you see that the "average" of $f(t)$ over one

[1] Use $f(t + 2\pi) = f(t)$ and $\cos(m(t + 2\pi)) = \cos(mt)$ and $\sin(m(t + 2\pi)) = \sin(mt)$.

period is $a_0/2$, because the cosine and sine functions have positive values and negative values which make them 0 on average. Therefore, we have

$$\frac{a_0}{2} = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt \quad \rightarrow \quad a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt \tag{15}$$

By comparing this with $a_m$ of (12), we can easily guess (13). Indeed, when $T = 2\pi$, the factor $1/\pi = 2/T$ comes correctly. Then, we can conclude (14) is also correct because the factor $\frac{2}{T}$ needs to be universal for both (13) and (14), as the factor $\frac{1}{\pi}$ is universal for both $a_m$ and $b_m$ in (12).

Instead of using sine and cosine functions, we can re-express all these results using complex numbers by considering Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. Using

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{16}$$

(1) can be re-expressed as

$$f(t) = \sum_{n=0}^{\infty}\left(A_n e^{\frac{2\pi i}{T}nt} + B_n e^{-\frac{2\pi i}{T}nt}\right) \tag{17}$$

where

$$\frac{1}{2}(a_n - ib_n) = A_n, \qquad \frac{1}{2}(a_n + ib_n) = B_n \tag{18}$$

Similarly, (13) and (14) become

$$A_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-\frac{2\pi i}{T}nt}dt, \qquad B_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{\frac{2\pi i}{T}nt}dt \tag{19}$$

(**Problem 4.** Show (17), (18), and (19)!) We can write all these relations even more compactly. Let

$$C_n \equiv A_n \quad \text{for } n > 0, \qquad C_0 \equiv A_0 + B_0 \tag{20}$$

$$C_n \equiv B_{-n} \quad \text{for } n < 0 \tag{21}$$

Then, (17) and (19) become

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{2\pi i}{T}nt} \tag{22}$$

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-\frac{2\pi i}{T}nt}dt \tag{23}$$

Fourier transformation can be generalized to the case of a non-periodic function if the period is allowed to become infinite (i.e., $T \to \infty$). In this case, the infinite sum in (22) is replaced with an integral. i.e.,

$$f(t) = \int_{-\infty}^{\infty} C_n e^{\frac{2\pi i}{T}nt}dn \tag{24}$$

Now, let

$$\omega \equiv \frac{2\pi}{T}n, \quad \tilde{f}(\omega) = \frac{T}{2\pi}C_n \tag{25}$$

then, (22) and (23) become

$$f(t) = \int_{-\infty}^{\infty} \tilde{f}(\omega)e^{i\omega t}d\omega \tag{26}$$

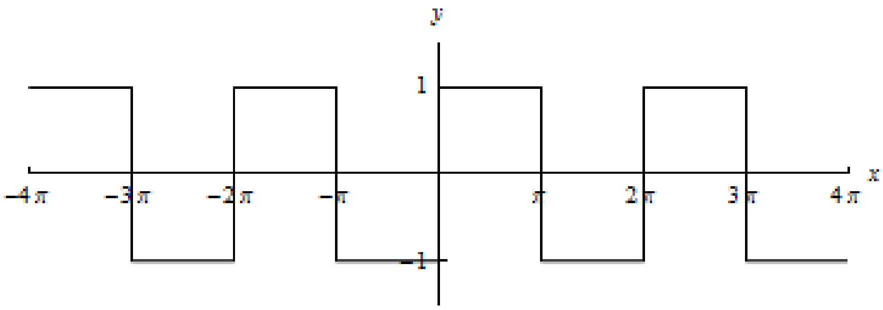
$$\tilde{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \tag{27}$$

So, this is very elegant. The Fourier transformation looks very symmetric! If you want, you can make it more symmetric by including $1/\sqrt{2\pi}$ factor in the right-hand side of (26). For example,

$$g(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{g}(\omega)e^{i\omega t}d\omega, \qquad \tilde{g}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt \tag{28}$$

As we will see, formulas (28) play a central role in quantum mechanics. There, we will also see that this symmetricity in Fourier transformation can be also obtained from complex conjugation. It is always amazing to see a same mathematical formula from different perspectives.

**Problem 4.** Find Fourier coefficients (i.e. (10) and (11)) for the following graph.




## Summary

- If

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nt) + b_n \sin(nt)\right]$$

the Fourier coefficients $a_n$ and $b_n$ are obtained by considering the integrations

$$\int_0^{2\pi} f(t)\cos(nt)dx \quad \text{and} \quad \int_0^{2\pi} f(x)\sin(nt)dt$$

- If

$$f(t) = \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t} d\omega$$

we have

$$\tilde{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$