An Introduction to General Relativity

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Abstract

This article should be accessible to students familiar with the Lagrangian formulation of classical mechanics, and special relativity except for the final sections that deal with black hole thermodynamics, which require some knowledge on thermodynamics and quantum field theory.

Contents

1 Introduction

Einstein's general relativity deals with gravitation. Given that Newton successfully described gravitation in the 17th century, why would one need another theory? Einstein discovered the theory of special relativity in 1905. General relativity was an attempt to describe gravitation in a way compatible with the theory of special relativity. Einstein achieved this goal in 1915.

Moreover, by Einstein's day, it was known that there was a phenomenon that Newtonian mechanics couldn't explain. The motion of Mercury around the Sun clearly deviated, even though by a very tiny amount, from what Newtonian mechanics predicted even when the gravitation due to other planets, in addition to that due to the Sun, was taken account. Einstein successfully explained this discrepancy in his paper on general relativity in 1915.

Furthermore, in the same paper, he calculated how much the light passing near the Sun would bend due to the Sun's gravitation using general relativity; he predicted that the light would bend double the angle that Newtonian mechanics predicted. (Remember that in Newtonian mechanics as well, any object passing by the Sun experiences the same acceleration regardless of its mass or speed.) Einstein's prediction was confirmed in 1919.

Now, let me briefly describe what the language of general relativity is like. To this end, let's briefly recall how Newton's law of gravity works. It states how much gravitational forces are exerted, given masses of objects and distances between them. If you know these forces, you will be able to calculate the trajectory of the objects by solving differential equations. On the other hand, Einstein's general relativity doesn't talk about forces at all but only the trajectory due to gravitation. This trajectory can be calculated if we know how much spacetime is curved, as one can easily imagine that the trajectory won't be straight if spacetime is curved. How much spacetime is curved (i.e. the curvature) in turn is expressed in terms of the metric, a construct that contains the information needed to calculate the distances between points. As spacetime is curved if a massive object is present, Einstein's equation gives the curvature of spacetime in terms of the mass of objects (precisely speaking, the energy-momentum tensor of objects). And, if we know the curvature of spacetime, we can obtain the metric and then calculate the trajectory.

The curvature, the energy-momentum tensor, and the metric are expressed in terms of mathematical objects called "tensors." We will describe what a tensor is and how it works in the first several sections of this paper. Then, in Section 12, we will obtain an equation for the trajectory given the metric of the spacetime. In the several sections following, we will obtain the explicit form of the curvature tensor in terms of the metric. Then, in Section 19, we will derive Einstein's equation by using the action principle (i.e.

using the Euler-Lagrange equation). In Section 22, we will show that the general theory of relativity reduces to Newtonian gravity in the appropriate limits, as it should if it is a correct theory that can describe all the phenomena Newton's law of gravity can explain. After explaining what a black hole is in Section 23, we will examine the two experimental successes of general relativity mentioned earlier, namely, the anomalous precession of the perihelion of Mercury and the bending of light. In Section 25, we will explain gravitational waves, of which the existence was experimentally proved much more recently. In Sections 26, 27, and 28, we explain Lie bracket, and Lie derivative without which a course on general relativity cannot be complete. In Sections 29 and 30, we will briefly discuss black hole thermodynamics, which deals with quantum aspects of black holes. This advanced topic is important but is left out of most general relativity textbooks. Then, we will provide a summary of the article. Readers are encouraged to consult the summary frequently while studying this article.

2 Metric tensor

The distance s between the point (x_1, y_1, z_1) and the point (x_2, y_2, z_2) can be calculated using Pythagoras' theorem:

$$s^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \tag{1}$$

which can be re-written as follows:

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{2}$$

When Δs is infinitesimal, (2) can be re-written as follows:

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{3}$$

What is the cylindrical coordinate version of (3)? To calculate this, let's recall the relation between the cylindrical coordinate and the Cartesian coordinate. It is:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned} \tag{4}$$

To write the cylindrical coordinate version of (3), let's calculate dx, dy, and dz in terms of dr, $d\theta$ and dz:

$$dx = dr\cos\theta - r\sin\theta d\theta$$
$$dy = dr\sin\theta + r\cos\theta d\theta$$
$$dz = dz \tag{5}$$

Plugging these relations to (3), we get

$$ds^2 = (dr\cos\theta - r\sin\theta d\theta)^2 + (dr\sin\theta + r\cos\theta d\theta)^2 + dz^2$$
$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 \tag{6}$$

Similarly, for spherical coordinates, we get

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{7}$$

where

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta \tag{8}$$

Now we are in a concrete position to understand what the metric tensor is. The metric tensor is defined by the following formula:

$$ds^2 = g_{ij}dx^i dx^j \tag{9}$$

where ds is the line element considered earlier, the dx^is are coordinates and the Einstein summation convention is used. For example, in the case of Cartesian coordinates, we have $(x^1, x^2, x^3) = (x, y, z)$ and the metric tensor is the identity matrix.

$$g_{ij} = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10}$$

Similarly, for cylindrical coordinates, we have $(x^1, x^2, x^3) = (r, \theta, z)$, and we get

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

For spherical coordinates, we have $(x^1, x^2, x^3) = (r, \theta, \phi)$, and we get

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & (x^1\sin x^2)^2 \end{pmatrix} \tag{12}$$

In the theory of special relativity, proper distances are invariant under Lorentz transformations. Proper distance Δs is defined by

$$\Delta s^2 = -c^2(t_2 - t_1)^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \tag{13}$$

If we let $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$, we get the following metric:

$$g_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{14}$$

In some textbooks, authors use the following convention for the metric.

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{15}$$

$$\Delta\tau^2 = g_{ij}\Delta x^i \Delta x^j \tag{16}$$

where $\Delta\tau$ is proper time.

Physicists usually use g to denote the determinant of the metric tensor (or, equivalently, the metric 'matrix') g_{ij}. How is the determinant of g related to the Jacobian? One can easily find the relation using the following argument. Let's consider the case that observer S and observer S' measure the distance using the coordinate system x and x', respectively. Since they should agree on the proper distance, we can write:

$$ds^2 = g'_{ij} dx'^i dx'^j = g_{ab} dx^a dx^b \tag{17}$$

$$= g'_{ij} \frac{\partial x'^i}{\partial x^a} dx^a \frac{\partial x'^j}{\partial x^b} dx^b \tag{18}$$

(Note that the ∂x^a in the 'denominator' counts as a lower index for the Einstein summation convention.) Here, we have used the following chain rule:

$$dx'^i = \frac{\partial x'^i}{\partial x^a} dx^a \tag{19}$$

Notice that this equation is the generalization of (5). Comparing (17) with (18), we obtain the following equation:

$$g_{ab} = g'_{ij} \frac{\partial x'^i}{\partial x^a} \frac{\partial x'^j}{\partial x^b} \tag{20}$$

Problem 1. A careful reader may notice that one of the formulas in our earlier article "Rotation and the Lorentz transformation, orthogonal and unitary matrices" is a special case of the above formula. Which one is it?

Notice also that the following are simply Jacobian matrices:

$$\frac{\partial x'^i}{\partial x^a}, \frac{\partial x'^j}{\partial x^b} \tag{21}$$

Therefore, equation (20) means that the matrix g_{ab} is the matrix multiplication of g'_{ij} and two Jacobian matrices. Taking the determinant on both sides of this equation and considering the case where the coordinate system x' is Cartesian (so $g'_{ij} = I$), we obtain

$$g \equiv \det g_{ab} = J^2 \tag{22}$$

where J is the Jacobian. Therefore, we obtain:

$$J = \sqrt{g} \tag{23}$$

We can use this result to express the integral of the volume element as follows:

$$V = \int dV = \int \sqrt{g} dx^1 dx^2 dx^3 = \int \sqrt{g} d^3x \tag{24}$$

where d^3x is the short hand notation of $dx^1 dx^2 dx^3$.

In the case of 4-dimensional Minkowski space, the determinant of the metric tensor is negative. For example, in (14), $\det g$ is -1. Therefore, $\sqrt{g}$ is not real. In this case, the Jacobian is given by $\sqrt{-g}$, as can be seen by taking the determinant of both sides of (20) when g'_{ij} is given by (15).

Problem 2. Why can we choose a symmetric (i.e. $g_{ab} = g_{ba}$) metric?

3 Vector, covector and inner-product

Given the metric g_{ij}, what is the length of the vector A^i? (A^i is a short hand notation for $(A^1, A^2, \cdots, A^n)$, i.e. i runs from 1 to n.) From (9), it is clear that this is given by the following formula:

$$|\overrightarrow{A}| = |A^i| = \sqrt{g_{ij} A^i A^j} \tag{25}$$

From the following convention,

$$|\overrightarrow{A}| = \sqrt{\overrightarrow{A} \cdot \overrightarrow{A}} \tag{26}$$

we can derive the following relation:

$$\overrightarrow{A} \cdot \overrightarrow{A} = g_{ij} A^i A^j \tag{27}$$

This suggests a more general definition for the dot product:

$$\overrightarrow{A} \cdot \overrightarrow{B} = g_{ij} A^i B^j \tag{28}$$

Using $\overrightarrow{A} \cdot \overrightarrow{B} = A^i B_i$, we write the inner-product of $\overrightarrow{A}$ and $\overrightarrow{B}$ as follows:

$$\overrightarrow{A} \cdot \overrightarrow{B} = g_{ij} A^i B^j = A^i B_i \tag{29}$$

which suggests the following definition:

$$g_{ij} B^j = B_i \tag{30}$$

The object B_i is called a 'covector.' While a vector, such as A^i, has one upper index, a covector has one lower index. We see here that one can lower the index of a vector by multiplying by the metric.

Now, I will define g^{ij}, the inverse of the metric matrix as follows:

$$g^{ij} g_{jk} = \delta^i_k \tag{31}$$

Here, δ^i_k is called the "Kronecker delta" and is a component notation for the identity matrix. δ^i_k is 1 if $k = i$, and 0 otherwise.

Let's multiply both sides of (30) by the inverse metric, to obtain:

$$\begin{aligned} g^{ki} g_{ij} B^j &= g^{ki} B_i \\ \delta^k_j B^j &= g^{ki} B_i \\ B^k &= g^{ki} B_i \end{aligned} \tag{32}$$

Thus, we see that one can raise the index of a covector by multiplying by the inverse of the metric. Finally, we have the following formula, whose proof is left as an exercise for the reader.

$$\overrightarrow{A} \cdot \overrightarrow{B} = g^{ij} A_i B_j \tag{33}$$

Problem 3. Evaluate $g^{ij} g_{ji}$ (Hint[1])

Problem 4. Obtain g^{ij} for (14).

[1]The Einstein summation convention is assumed.

4 Tensor and covariance

A tensor is a further generalization of scalars, vectors, and matrices. A rank (p, q) tensor is an object with p upper indices and q lower indices. A scalar has no indices. Therefore, it's a rank $(0, 0)$ tensor. A vector has one upper index. Therefore, it's a rank $(1, 0)$ tensor. A covector has one lower index. Therefore it's a rank $(0, 1)$ tensor. Similarly, a metric tensor (or, equivalently, a metric matrix) is a rank $(0, 2)$ tensor, and the inverse of the metric tensor is a rank $(2, 0)$ tensor.

A rank (p, q) tensor is also called a rank $(p + q)$ tensor, if the number of upper and lower indices is unimportant. It is easy to see that a matrix is a rank 2 tensor, because it has two indices. We have explained all this in our earlier article "The tensor."

Given this, let's give another interpretation of a (p, q) rank tensor. Using Einstein's summation convention we can understand a (p, q) rank tensor as a multilinear map that gives us a scalar if we feed in q vectors and p covectors. In a mathematical notation, this is:

$$c = T^{a_1 a_2 \cdots a_p}_{b_1 b_2 \cdots b_q} (A_1)^{b_1} (A_2)^{b_2} \cdots (A_q)^{b_q} (B_1)_{a_1} (B_2)_{a_2} \cdots (B_p)_{a_p} \tag{34}$$

where c is the scalar we obtain, the A's are vectors, and the B's are covectors.

Now, let's ask a question. How does tensor transform under a change of the coordinates? To answer this, let's consider the easy cases first. First of all, a scalar doesn't transform because the choice of coordinates does not affect the value of a scalar field. Second, the vector transforms as in (19):

$$A'^i = \frac{\partial x'^i}{\partial x^a} A^a \tag{35}$$

where A is a vector.

Given this, how should a covector transform? As an inner product between a vector and a covector is an invariant scalar, coordinate transformations of covectors should respect this property. By this argument, we can write the following:

$$\begin{aligned}
A'^i B'_i &= \frac{\partial x'^i}{\partial x^a} A^a B'_i = A^a B_a \\
\frac{\partial x'^i}{\partial x^a} B'_i &= B_a \\
\frac{\partial x^a}{\partial x'^j} \frac{\partial x'^i}{\partial x^a} B'_i &= \frac{\partial x^a}{\partial x'^j} B_a \\
\delta^i_j B'_i &= \frac{\partial x^a}{\partial x'^j} B_a \\
B'_j &= \frac{\partial x^a}{\partial x'^j} B_a
\end{aligned} \tag{36}$$

Inserting (35) and (36) to (34), and using the same trick as the steps leading to (36), we obtain:

$$T'^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} = \frac{\partial x'^{i_1}}{\partial x^{a_1}} \frac{\partial x'^{i_2}}{\partial x^{a_2}} \cdots \frac{\partial x'^{i_p}}{\partial x^{a_p}} \frac{\partial x^{b_1}}{\partial x'^{j_1}} \frac{\partial x^{b_2}}{\partial x'^{j_2}} \cdots \frac{\partial x^{b_q}}{\partial x'^{j_q}} T^{a_1 a_2 \cdots a_p}_{b_1 b_2 \cdots b_q} \tag{37}$$

If an object obeys this transformation property, we say that it transforms "covariantly." The property is called covariance. In particular, a tensor transforms covariantly. Notice that (20) shows that the metric transforms covariantly, so the metric tensor is indeed a tensor.

Finally, one can "raise" the tensor indices and "lower" the tensor indices using the metric tensor. For example, one can write the following:

$$R_{abcd} = R_{abc}{}^{e} g_{ed} \tag{38}$$

$$R_{abcd} g^{de} = R_{abc}{}^{e} \tag{39}$$

where $R_{abc}{}^{e}$ is a $(1,3)$ rank tensor, and R_{abcd} is a rank $(0,4)$ tensor. The above equations are generalizations of (30) and (32).

Problem 5. Show that C^{ab}, the rank-2 tensor obtained by multiplying two rank-1 tensors A^a, B^b as follows, transforms covariantly:

$$C^{ab} = A^a B^b$$

This multiplication is called a "tensor product." We can take the tensor product of any two tensors, not just two rank-1 tensors. For example,

$$F^{abcde} = D^{ab} E^{cde}$$

Problem 6. Let T^a_b given by the Kronecker-delta as $T^a_b = \delta^a_b$. Then, show that, under the coordinate transformation of (37), T'^i_j is also given by the Kronecker-delta as $T'^i_j = \delta^i_j$.

Problem 7. Let's say that S represents a Cartesian coordinate system, and S' represents another Cartesian coordinate system, obtained by rotating S by θ degree about z-axis. Then, it is easy to see that a vector transforms as follows:

$$\begin{aligned} V'_x &= V_x \cos\theta + V_y \sin\theta \\ V'_y &= -V_x \sin\theta + V_y \cos\theta \\ V'_z &= V_z \end{aligned}$$

Given this, show how the components of a tensor T'_{xx}, T'_{xy}, T'_{zz} are explicitly given in terms of Ts observed in S.

5 Electromagnetic tensor

In this section, we give an example of a tensor, the electromagnetic tensor, and its transformation, since the discussion in the previous section could sound too abstract without an explicit example.

The electromagnetic field $F_{\mu\nu}$, which is a rank-2 tensor, is defined as follows:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{40}$$

Notice that it is totally anti-symmetric. (i.e. $F_{\nu\mu} = -F_{\mu\nu}$) The "four-current," a rank-1 tensor, which combines the charge and current density into one entity, is defined as follows:

$$J^\nu = (c\rho, j_x, j_y, j_z) \tag{41}$$

With these definitions, one can write Maxwell's equations in the following simple forms:

$$\partial^\mu F_{\mu\nu} = J_\nu \tag{42}$$

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 \tag{43}$$

By explicit calculation one can check (42) leads to

$$\nabla \cdot \vec{E} = \rho \tag{44}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j} \tag{45}$$

while (43) leads to

$$\nabla \cdot \vec{B} = 0 \tag{46}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{47}$$

Now, we ask the following question: How would the electromagnetic field and the four-current transform under Lorentz transformation? The answer for the four-current is easy, since it's a four-vector. Namely, if we write the Lorentz transformation matrix as follows:

$$\Lambda^{\nu'}{}_{\mu} \equiv \begin{pmatrix} \gamma & 0 & 0 & -\gamma\frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\frac{v}{c} & 0 & 0 & \gamma \end{pmatrix} \tag{48}$$

then we have:

$$J^{\nu'} = \Lambda^{\nu'}{}_{\mu} J^{\mu} \tag{49}$$

Similarly, in light of (37), (i.e. $\Lambda^{\nu'}{}_{\mu} = \partial x'^{\nu}/\partial x^{\mu}$) we have the following:

$$F_{\alpha'\beta'} = \Lambda_{\alpha'}{}^{\mu} \Lambda_{\beta'}{}^{\nu} F_{\mu\nu} \tag{50}$$

where $\Lambda_{\alpha'}{}^{\mu}$ is the inverse of $\Lambda^{\mu}{}_{\alpha'}$

The transformations (49) and (50) are consistent with Maxwell's equations (i.e. (42) and (43)), since:

$$\begin{aligned}
\partial^{\alpha'} F_{\alpha'\beta'} &= \partial^{\rho} \Lambda_{\rho}{}^{\alpha'} (\Lambda_{\alpha'}{}^{\mu} \Lambda_{\beta'}{}^{\nu} F_{\mu\nu}) && (51)\\
&= \partial^{\rho} \delta^{\mu}_{\rho} F_{\mu\nu} \Lambda_{\beta'}{}^{\nu} && (52)\\
&= \partial^{\mu} F_{\mu\nu} \Lambda_{\beta'}{}^{\nu} = J_{\nu} \Lambda_{\beta'}{}^{\nu} = J_{\beta'} && (53)\\
\partial^{\alpha'} F_{\alpha'\beta'} &= J_{\beta'} && (54)
\end{aligned}$$

and similarly for (43). In other words, the covariant character of Maxwell's equations makes it manifest that they be satisfied under any reference frame.

As an aside, notice that (42) implies:

$$\begin{aligned}
\partial^{\nu} J_{\nu} &= \partial^{\nu} \partial^{\mu} F_{\mu\nu} && (55)\\
&= \partial^{\nu} \partial^{\mu} \{\frac{1}{2}(F_{\mu\nu} - F_{\nu\mu})\} && (56)\\
&= \frac{1}{2} \partial^{\nu} \partial^{\mu} F_{\mu\nu} - \frac{1}{2} \partial^{\mu} \partial^{\nu} F_{\nu\mu} && (57)\\
&= \frac{1}{2} \partial^{\nu} \partial^{\mu} F_{\mu\nu} - \frac{1}{2} \partial^{\nu} \partial^{\mu} F_{\mu\nu} && (58)\\
&= 0 && (59)
\end{aligned}$$

where, from the first line to the second line, we used the fact that $F_{\nu\mu} = -F_{\mu\nu}$ and, from the third line to the fourth line, we used the fact that μ and ν are dummy indices in Einstein summation convention and thus can be replaced by ν and μ. In conclusion, $\partial^{\nu} J_{\nu} = \partial_{\mu} J^{\mu} = 0$ implies the conservation of charge, since it yields the following:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0 \tag{60}$$

Now some historical remarks. Lorentz more or less successfully derived (50) in 1904, even though he didn't use the tensor notation but instead the component by component formulation as follows (in modern notation):

$$E'_x = E_x \qquad E'_y = \gamma(E_y - vB_z) \qquad E'_z = \gamma(E_z + vB_y) \tag{61}$$

$$B'_x = B_x \qquad B'_y = \gamma(B_y + \frac{v}{c^2} E_z) \qquad B'_z = \gamma(B_z - \frac{v}{c^2} E_y) \tag{62}$$

However, he didn't get (49) right, so he failed in exactly reproducing (54). In 1905, Einstein successfully fixed this problem.

Problem 8. Let's say that we have two vector fields $V^\mu(x)$ and $U^\nu(x)$ that transform covariantly. Show that a new vector field $S^\nu \equiv V^\mu \partial_\mu U^\nu - U^\mu \partial_\mu V^\nu$ transforms covariantly as well.

6 Levi-Civita symbol

The Levi-Civita symbol $\tilde{\epsilon}_{a_1 a_2 a_3 \cdots a_n}$ is defined by the following two conditions:

$$\tilde{\epsilon}_{123\cdots n} = 1 \tag{63}$$

$$\tilde{\epsilon}_{\cdots i \cdots j \cdots} = -\tilde{\epsilon}_{\cdots j \cdots i \cdots} \tag{64}$$

In other words, the Levi-Civita symbol picks up an extra negative sign when two indices are exchanged. Notice that the Levi-Civita symbol is zero if the same index is repeated. This is easy to see as follows.

$$\tilde{\epsilon}_{\cdots i \cdots i \cdots} = -\tilde{\epsilon}_{\cdots i \cdots i \cdots} \tag{65}$$

where we have replaced j in (64) by i. So we conclude:

$$\tilde{\epsilon}_{\cdots i \cdots i \cdots} = 0 \tag{66}$$

Therefore, the Levi-Civita symbol can take -1, 0 and 1 as its value but never any other number. As a side note, in general relativity, for which the indices usually run from 0 to $n-1$, where n is the dimension of spacetime, rather than 1 to n, the Levi-Civita symbol is defined similarly but with the first condition (63) replaced by the following:

$$\tilde{\epsilon}_{012\cdots n-1} = 1 \tag{67}$$

Given this, we ask the following question: Is the Levi-Civita symbol a tensor?

If Levi-Civita symbol is a tensor, if we contract it with tensors, it should still be a tensor. Let's check whether this is true. Let's contract the Levi-Civita symbol with the inverse of the metric tensor as follows:

$$\tilde{\epsilon}_{a_1 a_2 a_3 \cdots a_n} \tilde{\epsilon}_{b_1 b_2 b_3 \cdots b_n} g^{a_1 b_1} g^{a_2 b_2} \cdots g^{a_n b_n} \tag{68}$$

$$= n!(\det g^{ab}) = n!(\det g_{ab})^{-1} = n! g^{-1} \tag{69}$$

where we used the fact that g^{ab} is the matrix inverse of g_{ab}, at the second equality in the second line.

Since (68) has no free indices (i.e. all indices are dummy indices which are contracted) it should be a rank-0 tensor (i.e. a scalar), if the Levi-Civita symbol is a tensor.

Remember that a scalar is a quantity that doesn't depend on the choice of the coordinate system. (If you don't understand this point, please look at (37). In the case of a scalar, (37) will just be $T' = T$.) However, (69) clearly shows (68) is not a scalar, as the determinant of the metric tensor depends on how you choose the coordinate system.

Problem 9. Prove $T'^i_i = T^a_a$ using (37). (Hint[2]) This implies that T^a_a is a scalar as it doesn't transform under a change of the coordinates. One could immediately arrive at the same conclusion by noticing that it has no free indices which implies it should be a rank-0 tensor (i.e. a scalar). That the trace of a matrix is a scalar plays an important role later in group theory.

7 Levi-Civita tensor

Given this situation, is there anyway that we can modify the Levi-Civita symbol to become a true tensor? To this end, let's define the Levi-Civita tensor $\epsilon_{a_1a_2a_3\cdots a_n}$ as follows:

$$\epsilon_{a_1a_2a_3\cdots a_n} = \sqrt{|g|}\tilde{\epsilon}_{a_1a_2a_3\cdots a_n} \tag{70}$$

Let's check whether this is a tensor. Let's do the same trick that we did in Section 6. We have:

$$\epsilon_{a_1a_2a_3\cdots a_n}\epsilon_{b_1b_2b_3\cdots b_n}g^{a_1b_1}g^{a_2b_2}\cdots g^{a_nb_n} \tag{71}$$

$$= |g|(n!g^{-1}) = \pm n! \tag{72}$$

where the positive sign is taken when g is positive and the negative sign is taken when g is negative. Unlike the case of the Levi-Civita symbol, we see that the Levi-Civita tensor is a true tensor as (72) is a scalar that doesn't depend on the choice of coordinate system. So, we can justify the name "Levi-Civita tensor."

Given this, we can raise the indices of the Levi-Civita tensor. For example,

$$\epsilon^{a_1a_2\cdots a_m}{}_{b_{m+1}b_{m+2}\cdots b_n} = g^{a_1b_1}g^{a_2b_2}\cdots g^{a_mb_m}\epsilon_{b_1b_2\cdots b_n} \tag{73}$$

In particular notice the following:

$$\epsilon^{12\cdots n} = \pm\frac{1}{\sqrt{|g|}} \tag{74}$$

where the sign notation is as before.

As an aside, I want to note that my paper with Brian Kong "Black hole entropy predictions without Immirzi parameter and Hawking radiation of single-partition black hole" is based on the observation that traditional loop quantum gravity used the Levi-Civita symbol where the Levi-Civita tensor should be used in defining the area operator

[2]Try to prove $T'^i_i = \delta^b_a T^a_b$.

i.e. an operator that gives you the possible values for area by its eigenvalues as they are quantized in loop quantum gravity. Therefore, our paper pointed out that the traditional formula that gives you the area operator in terms of the metric was not correct, and must be given by a new formula.

8 The equation of motion in field theory

In the earlier article, "the Lagrangian formulation of classical mechanics," we have seen how one can derive the equation of motion if Lagrangian is given. We will derive the analogous field theory version of the equation of motion in this section.

The basic idea is same. The only difference is that action is given in terms of Lagrangian density, which gives action, when integrated over spacetime. Remember that in classical mechanics, the action is given in terms of Lagrangian, which gives action, when integrated over time. Therefore, in field theory, the action can be written as follows.

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \tag{75}$$

where ∂_μ are partial derivatives with respect to spacetime: $(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$

Then, we have:

$$\begin{aligned} 0 &= \delta S = \int d^4x \{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \} && (76) \\ &= \int d^4x \{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) \} && (77) \end{aligned}$$

where Einstein summation convention is used. (μ is the dummy variable.)

Since as exactly in the case of classical mechanics, the last term vanishes because it's a total derivative term and we set $\delta\phi = 0$ at infinity. Thus, noticing that δS should be zero for an arbitrary $\delta\phi$, we conclude that the equation of motion is given by:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \tag{78}$$

Compare this with classical mechanics case:

$$\frac{\partial L}{\partial \phi} - \partial_t \left(\frac{\partial L}{\partial(\partial_t \phi)} \right) = 0 \tag{79}$$

Everything is same, except for the fact that in field theory case we have the spacetime derivative instead of the time derivative, and we have the Lagrangian density instead of the Lagrangian.

9 A brief preview of Einstein's theory of general relativity

Having introduced the necessary basic ingredients, we now give a lightning preview of Einstein's theory of general relativity. To fully understand the material, one should read many more sections of this article.

The key idea of the theory of general relativity is that the Lagrangian density due to the curvature of the spacetime background is proportional to the curvature scalar R (also called the Ricci scalar). The curvature scalar R is given by $g^{ab}R_{ab}$ where R_{ab} is the Ricci tensor, which contains the information about in which direction and how much spacetime is curved. The Ricci tensor is given in terms of the metric, since the metric can tell you all this information. For example, it is apparent that a curved space cannot be described by Cartesian coordinates if the metric tensor is the identity matrix as in (10); the space would be totally flat at every point. Certainly, you cannot measure the distance from Seoul to Boston by placing a ruler on a world map.

There are three terms in the general relativistic Lagrangian density: a curvature Lagrangian density, a vacuum Lagrangian density (the cosmological constant term), and a matter Lagrangian density. (The Lagrangian matter density contains any other fields under consideration that are not expressible by the metric. If you have learned some quantum field theory, you may have learned how the Lagrangian matter densities for Klein-Gordon fields and Dirac fields are given. Nevertheless, their explicit forms won't play any roles in this article, other than that we get the energy-momentum tensor if we differentiate the matter Lagrangian density with respect to the metric.) In the Euler-Lagrange equation, the metric is varied since the Lagrangian density is given by it.

The curvature action is

$$\int d^4x\sqrt{-g}\left(\frac{R}{16\pi G}\right) \tag{80}$$

where g denotes the determinant of the metric and R denotes the curvature scalar. The origin of the factor $16\pi G$ in the denominator will be clear when we talk about Einstein field equation in Newtonian limit in Section 22. As observed at the end of Section 2, $\sqrt{-g}$ is the Jacobian; if you integrate this Jacobian, you get a volume. It is natural that the integrand contain the volume form, and indeed, all three terms of the Lagrangian contain this factor. The vacuum action (more precisely, the action for the vacuum energy) is simple because it is proportional to the volume only:

$$\int d^4x\sqrt{-g}\left(\frac{-2\Lambda}{16\pi G}\right) \tag{81}$$

where Λ is a constant called the cosmological constant, and the factor of -2 is used by

convention. The action for matter is simply

$$\int d^4x\sqrt{-g}\mathcal{L}_m \tag{82}$$

where $\mathcal{L}_m$ denotes the Lagrangian density of matter. Given this, we are now ready to obtain Einstein's equation, equivalently, the equation of motion or the Euler-Lagrange equation. Summing up these three terms, differentiating their sum with respect to the metric, and setting the result equal to zero, we get Einstein's equation. (There is actually a subtlety here. Rigorously speaking, for this Euler-Lagrange equation, there should be contributions coming from the spacetime derivatives of the metric, but it turns out that they don't play any roles. We will justify this point in Section 19) When differentiating, you have to use the useful fact that

$$\frac{\partial g}{\partial g^{ab}} = -gg_{ab} \tag{83}$$

A derivation of (83) can be found on Appendix A. Instead of (83), one may equivalently use the following relation:

$$\delta(g^{-1}) = \frac{1}{g}g_{ab}\delta g^{ab} \tag{84}$$

where δ denotes the variation. To derive this, first note the following:

$$\mathrm{Tr}(\ln A) = \ln(\det A) \tag{85}$$

where A is a matrix. Given this, consider the following:

$$\begin{aligned}
\delta(\ln(\det M)) &= \ln(\det(M+\delta M)) - \ln(\det M) && (86)\\
&= \ln\left(\frac{\det(M+\delta M)}{\det M}\right) && (87)\\
&= \ln(\det(1+M^{-1}\delta M)) && (88)\\
&= \mathrm{Tr}(\ln(1+M^{-1}\delta M)) && (89)\\
&= \mathrm{Tr}(M^{-1}\delta M) && (90)
\end{aligned}$$

where from (88) to (89) we used (85). If we re-express the left-hand side, we obtain

$$\frac{1}{\det M}\delta(\det M) = \mathrm{Tr}(M^{-1}\delta M) \tag{91}$$

If we let $M = g^{ab}$, then we get $\det M = g^{-1}$ as $g = \det g_{ab}$. Plugging these relations into the above formula we obtain (84).

Now, we can just plug in:

$$\begin{aligned}
\delta\sqrt{-g} &= \delta[(-g^{-1})^{-1/2}]\\
&= -\frac{1}{2}(-g^{-1})^{-3/2}\delta(-g^{-1})\\
&= -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab} && (92)
\end{aligned}$$

Solving for the Euler-Lagrange equation, we are left with Einstein's equation:

$$R_{ab} - \frac{1}{2} g_{ab} R + g_{ab} \Lambda = 8\pi G T_{ab} \tag{93}$$

Here, T_{ab} denotes the energy momentum tensor, which is a $(0, 2)$ tensor obtained by differentiating the matter Lagrangian with respect to the metric as in (185). It has the property that T_{00} is the energy density and T_{0i} is the ith component of the momentum density.

If you are not sure how to obtain (93), look at (183) and (184). In many cases, the cosmological constant term in this equation can be set to zero, as it is very small.

In summary, when there is matter present, the right-hand side of Einstein's equation is determined, so solving Einstein's equation enables you to obtain R_{ab}, the curvature tensor. Once you know R_{ab}, you will know how much the spacetime is curved, which enables you to calculate the path taken by small test particles in this curved background.

10 Are partial derivatives of vector field covariant?

It is easy to see that the partial derivative of a scalar field is a covariant covector. Let's consider its coordinate transformation as follows:

$$\frac{\partial \phi}{\partial x'^i} = \frac{\partial x^a}{\partial x'^i} \frac{\partial \phi}{\partial x^a} \tag{94}$$

Comparing the above equation with (36), we conclude that $\partial_a \phi$ (a short-hand notation for the derivative of ϕ with respect to x^a) is a covector. Now, let's consider whether the partial derivatives of a vector field A transforms covariantly. The transformation law for vectors yields:

$$\begin{aligned} \partial'_i A'^j &= \frac{\partial}{\partial x'^i} \left(\frac{\partial x'^j}{\partial x^a} A^a \right) \\ &= \frac{\partial x^b}{\partial x'^i} \frac{\partial}{\partial x^b} \left(\frac{\partial x'^j}{\partial x^a} A^a \right) \\ &= \frac{\partial x^b}{\partial x'^i} \frac{\partial x'^j}{\partial x^a} \partial_b A^a + \frac{\partial x^b}{\partial x'^i} \frac{\partial^2 x'^j}{\partial x^b \partial x^a} A^a \end{aligned} \tag{95}$$

We note here that the presence of the second term in (95) prevents the partial derivative of a vector field from being covariant. This means that it is a coordinate-dependent construction. In the next section, we will introduce a new notion of derivative that remedies this situation.

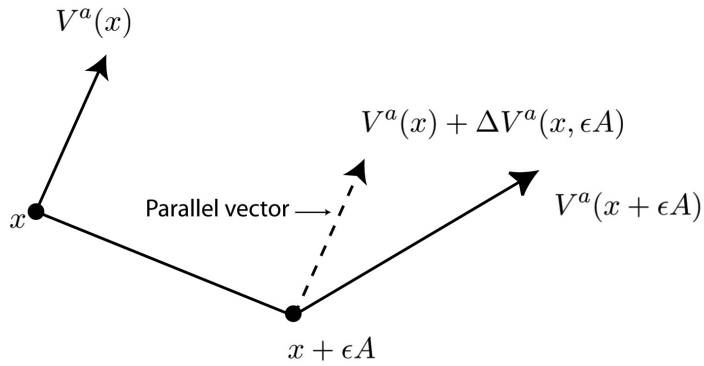

Figure 1: Comparing vector field at two neighboring points

11 Covariant derivative and the Christoffel symbol

The gradient of a scalar field tells you how rapidly the field changes as you move around in the space. It compares the value of the field at a point with that of a neighboring point and quantifies how much they differ. For example, the value of ϕ at two neighboring points (x) and $(x + \epsilon A)$ can be compared using the gradient covector $\partial_c \phi$ as follows.

$$A^c \partial_c \phi = \lim_{\epsilon \to 0} \frac{\phi(x + \epsilon A) - \phi(x)}{\epsilon} \tag{96}$$

Can we generalize this notion to a vector field, i.e. can we say how rapidly a vector field changes as you move around in space? What is a gradient of a vector field?

There is a difficulty inherent in this question. How can we compare a vector based at one point with a vector based at another point? We can always compare vectors based at the same point, since the tangent space forms a vector space, but comparing two vectors at two different points is more complicated.

Nevertheless, there is a way of comparing vector field at two neighboring points which leads to a natural definition of the "gradient" of a vector field. We need to specify which vectors are "parallel," even when they are based at different points. The vector V^a at the point $(x + \epsilon A)$ is compared with the vector $V^a(x) + \Delta V^a$, which is parallel to V^a at the point x (see Figure 1). Assuming a suitable definition for Δ, we define the "gradient" of a vector field $\nabla_c V^a$ as follows:

$$A^c \nabla_c V^a = \lim_{\epsilon \to 0} \frac{V^a(x + \epsilon A) - (V^a(x) + \Delta V^a(x, \epsilon A))}{\epsilon} \tag{97}$$

Derivatives such as ∇ are called covariant derivatives, as $\nabla_c V^a$ transforms covariantly.

In the above expression, we can see again why defining a gradient of a vector field by simple partial derivatives of the vector field components would not make sense. Aside

from the fact that the partial derivative does not transform covariantly, the important piece ΔV^a is ignored. Comparing the V^a's at two neighboring points without considering ΔV^a would be meaningless, as the coordinate systems are not necessarily flat Cartesian. For example, let's consider the case in which the vector field is given by $\hat{r}$ in polar coordinates. In components, the field is given by $(1, 0)$; the first component corresponding to $\hat{r}$ and the second component corresponding to $\hat{\theta}$. If we ignore the ΔV peace, we may wrongly conclude that the gradient of this vector field is zero, as differentiating $(1, 0)$ with respect to each coordinate yields zero. Since $\hat{r}$ rotates as you vary θ, the gradient should be nonzero. This is why ΔV is important.

Now, we must define ΔV^a in terms of V and ϵ. It is easy to see that ΔV should be linear in V. If we double the V, the parallel vector $V + \Delta V$ is doubled, and therefore, the difference between V and $V + \Delta V$ is doubled. As ϵ is taken to be infinitesimal, we are only interested in first order perturbations, so ΔV should be linear in ϵA as well. Clearly, when ϵA is zero, ΔV is zero. Therefore, we can express ΔV as follows:

$$\Delta V^a(x) = -\Gamma^a_{bc}(x) V^b(x) \epsilon A^c \tag{98}$$

Here, the expressions $\Gamma^a_{bc}(x)$ are collectively called the Christoffel symbol. The minus sign in front of the Christoffel symbol is a convention. Plugging the above equation into (97) yields the following equation:

$$\nabla_c V^a = \partial_c V^a + \Gamma^a_{bc} V^b \tag{99}$$

Later in this article, we will find Γ^a_{bc} explicitly, in terms of the metric. The Christoffel symbol encodes all the information necessary to compare vectors based at different points.

12 Geodesics

Let's take a break from tensors and consider geodesics. A geodesic is the path that extremizes the distance (or the proper time, if the path considered is 4-dimensional) between two points. For example, if you fly from Seoul to Boston (ignoring the wind speed and other effects), you want to fly along a geodesic because it minimizes the distance and time traveled. Furthermore, you may already know Fermat's principle which states that light moves along a trajectory that minimizes the time it takes to travel. Given these examples, Einstein guessed that particles move along geodesics in the presence of a gravitational field, which turned out to be correct. Now, let's calculate the equation for geodesics, given a metric g_{ab}.

The proper time that it takes to travel between two points P_1 and P_2 (or the length of a path between two points P_1 and P_2 if the path considered is 3-dimensional) is given by

$$\tau = \int_{P_1}^{P_2} d\tau = \int_{P_1}^{P_2} \sqrt{g_{bc}(x) dx^b dx^c} = \int_{P_1}^{P_2} \sqrt{g_{bc}(x) \frac{dx^b}{du} \frac{dx^c}{du}} du \tag{100}$$

so we want to extremize this quantity. Interestingly enough, we already know how to solve this problem. We know how to extremize the action S, given a Lagrangian L. Recall that the action is given by the following equation:

$$S = \int L(x^i, \dot{x}^i) dt \tag{101}$$

In this case, we know that the phase space trajectory that extremizes the action is characterized by the Euler-Lagrange equation:

$$\frac{\partial L}{\partial x^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = 0 \tag{102}$$

Equation (100) is precisely in the form of (101), except for the fact that dt (time differential) is replaced by du (path parameter differential).

Therefore, if we set

$$L = \sqrt{g_{bc}(x) \dot{x}^b \dot{x}^c} \tag{103}$$

where $\dot{x}^a = \frac{dx^a}{du}$, we can conclude that the geodesic will obey the following Euler-Lagrange equation:

$$\frac{\partial L}{\partial x^a} - \frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = 0 \tag{104}$$

Assuming that L is non-zero, we can multiply both sides by $-2L$:

$$\begin{aligned} 2L\left[\frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a}\right] &= 0 \\ \frac{d}{du}\left(\frac{\partial L^2}{\partial \dot{x}^a}\right) - \frac{\partial L^2}{\partial x^a} &= 2\frac{\partial L}{\partial \dot{x}^a}\frac{dL}{du} \end{aligned} \tag{105}$$

Plugging (103) into the left-hand side of the above equation yields

$$\begin{aligned} \frac{d}{du}\left(\frac{\partial L^2}{\partial \dot{x}^a}\right) - \frac{\partial L^2}{\partial x^a} &= \frac{d}{du}\left[\frac{\partial}{\partial \dot{x}^a}(g_{bc}\dot{x}^b\dot{x}^c)\right] - \frac{\partial}{\partial x^a}(g_{bc}\dot{x}^b\dot{x}^c) \\ &= \frac{d}{du}(2g_{ab}\dot{x}^b) - (\partial_a g_{bc})\dot{x}^b\dot{x}^c \\ &= 2g_{ab}\ddot{x}^b + 2\partial_c g_{ab}\dot{x}^b\dot{x}^c - \partial_a g_{bc}\dot{x}^b\dot{x}^c \\ &= 2g_{ab}\ddot{x}^b + 2\dot{x}^b\dot{x}^c\left[\frac{1}{2}(\partial_c g_{ba} + \partial_b g_{ca} - \partial_a g_{bc})\right] \end{aligned} \tag{106}$$

where in the last line we have used the following fact, which amounts to changing dummy indices:

$$\partial_c g_{ab}\dot{x}^b\dot{x}^c = \partial_b g_{ac}\dot{x}^c\dot{x}^b = \partial_b g_{ac}\dot{x}^b\dot{x}^c \tag{107}$$

On the other hand, plugging (103) into the right-hand side of (105) yields:

$$\begin{aligned} 2\frac{\partial L}{\partial \dot{x}^a}\frac{dL}{du} &= 2\frac{\partial\sqrt{g_{bc}\dot{x}^b\dot{x}^c}}{\partial \dot{x}^a}\frac{d}{du}\left(\frac{d\tau}{du}\right) \\ &= \frac{2g_{ad}\dot{x}^d}{\sqrt{g_{bc}\dot{x}^b\dot{x}^c}}\ddot{\tau} \\ &= 2(\ddot{\tau}/\dot{\tau})g_{ab}\dot{x}^b \end{aligned} \tag{108}$$

where $d\tau$ is defined in (100). Putting the sides together, we conclude:

$$g_{ab}\ddot{x}^b + \frac{1}{2}(\partial_c g_{ba} + \partial_b g_{ca} - \partial_a g_{bc})\dot{x}^b\dot{x}^c = (\ddot{\tau}/\dot{\tau})g_{ab}\dot{x}^b \tag{109}$$

Multiplying both sides by g^{ad} yields:

$$\ddot{x}^d + g^{ad}\frac{1}{2}(\partial_c g_{ba} + \partial_b g_{ca} - \partial_a g_{bc})\dot{x}^b\dot{x}^c = (\ddot{\tau}/\dot{\tau})\dot{x}^d \tag{110}$$

Renaming the indices,

$$\ddot{x}^a + g^{ad}\frac{1}{2}(\partial_c g_{bd} + \partial_b g_{cd} - \partial_d g_{bc})\dot{x}^b\dot{x}^c = (\ddot{\tau}/\dot{\tau})\dot{x}^a \tag{111}$$

If we let the parameter u be the proper time τ (since we can parametrize the path any way we want), we get $\dot{\tau} = 1$ and $\ddot{\tau} = 0$:

$$\frac{d^2x^a}{d\tau^2} + \frac{1}{2}g^{ad}(\partial_c g_{bd} + \partial_b g_{cd} - \partial_d g_{bc})\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0 \tag{112}$$

This is the equation of motion for the path of a particle that extremizes proper time.

13 Parallel transport along geodesic

Recall Figure 1, which explains the notion of parallel transport. Parallel transporting a vector along the straight path from (x) to $(x + \epsilon A)$ is expressed by:

$$V^a(x + \epsilon A) = V^a(x) + \Delta V^a(x, \epsilon A) \tag{113}$$

See Figure 2. Plugging the above equation into (97) we get:

$$A^c\nabla_c V^a = 0 \tag{114}$$

Now, suppose we parallel transported the unit vector tangent to the geodesic path (see Figure 3). Intuitively, we are moving in a straight line, turning neither left nor right. If we call the unit vector which points in the direction that we are moving V^c, we simply replace A^c by V^c in the above equation to obtain:

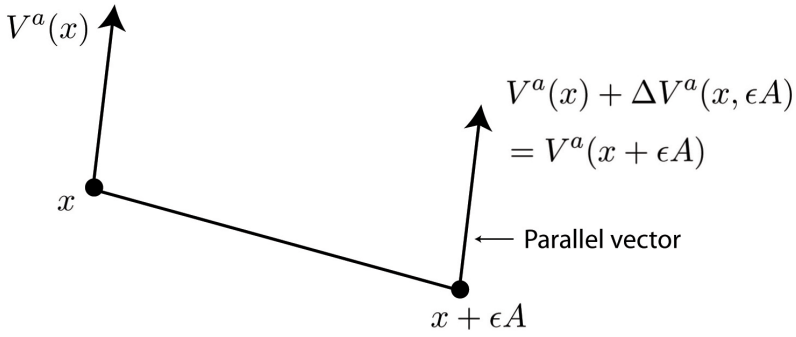

Figure 2: Parallel transporting a vector

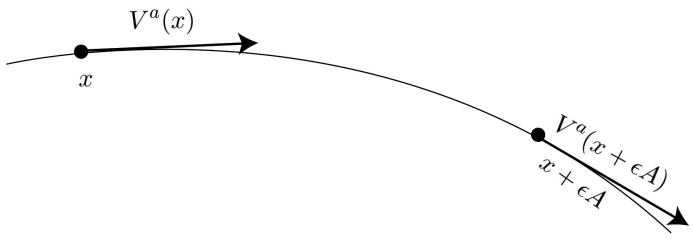

Figure 3: Parallel transporting the unit vector tangent to the geodesic path

$$V^c \nabla_c V^a = 0 \tag{115}$$

where V^a is given by

$$V^a = \frac{dx^a}{ds} \tag{116}$$

which is a unit vector because its norm can be computed as follows:

$$\begin{aligned} ds &= \sqrt{g_{ab} dx^a dx^b} \\ 1 &= \sqrt{g_{ab} \frac{dx^a}{ds} \frac{dx^b}{ds}} \end{aligned} \tag{117}$$

Plugging (116) and (99) into (115) yields the following:

$$\begin{aligned} \frac{dx^c}{ds} \left[\frac{\partial}{\partial x^c} \left(\frac{dx^a}{ds} \right) + \Gamma^a_{bc} \frac{dx^b}{ds} \right] &= 0 \\ \frac{d^2 x^a}{ds^2} + \Gamma^a_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} &= 0 \end{aligned} \tag{118}$$

Comparing the above formula with (112), we conclude that the Christoffel symbol is given by the following formula:

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_c g_{bd} + \partial_b g_{cd} - \partial_d g_{bc}) \tag{119}$$

if it is symmetric with respect to the lower two indices as follows:

$$\Gamma^a_{bc} - \Gamma^a_{cb} = 0 \tag{120}$$

The left-hand side of the above equation is called the torsion. When the above equation is true, the geometry associated with the Christoffel symbol is said to be "torsion-free." In this article, we will not delve much into the case of finite torsion, except for mentioning it briefly in Section 26. After all, Einstein's theory of general relativity was first formulated in a torsion-free geometry, and physics students don't usually study geometry with torsion in a standard general relativity course.

Problem 10. Obtain $\Gamma^r_{r\theta}$, $\Gamma^\theta_{\theta\phi}$, $\Gamma^\theta_{\phi\phi}$, and $\Gamma^\phi_{\theta\phi}$ for (12).

14 Covariant derivative of covectors and other tensors

So far, we have only considered the covariant derivative of vectors, but this construction can be generalized to aribtrary tensors. In this section, we will derive an expression for the covariant derivative of a general tensor in terms of the Christoffel symbol.

First, let us consider the rank-0 tensor. In Section 10, we saw that the partial derivative of a scalar field was covariant, so we make the natual definition:

$$\nabla_a \phi = \partial_a \phi \tag{121}$$

Given this, how should we define the covariant derivative of a covector? I propose the following general formula:

$$\nabla_c B_a = \partial_c B_a + \tilde{\Gamma}^b_{ac} B_b \tag{122}$$

Of course, there is no reason for $\tilde{\Gamma}$ to be equal to Γ. We will actually derive this quantity using (121).

However, before deriving this quantity, we will first assume that the covariant derivative satisfies Leibniz rule. For example,

$$\nabla_a (U^b V^{cd}) = (\nabla_a U^b) V^{cd} + U^b \nabla_a V^{cd}. \tag{123}$$

Leibniz rule is a key property that a derivative operator satisfies. Therefore, it is natural to assume that the covariant derivative satisfies it if it is a mathematically sound derivative operator.

A mathematical way of saying this is that we *require* the covariant derivative to follow Leibniz rule, as one of its *definition* as the covariant derivative.

Given this, let's begin our derivation. Since a vector and a covector can be contracted to form a scalar, we can write:

$$\begin{aligned}
\nabla_c (A^a B_a) &= \partial_c (A^a B_a) \\
(\nabla_c A^a) B_a + A^a (\nabla_c B_a) &= (\partial_c A^a) B_a + A^a (\partial_c B_a) \\
(\partial_c A^a + \Gamma^a_{bc} A^b) B_a + A^a (\partial_c B_a + \tilde{\Gamma}^b_{ac} B_b) &= (\partial_c A^a) B_a + A^a (\partial_c B_a) \\
\Gamma^a_{bc} A^b B_a + A^a \tilde{\Gamma}^b_{ac} B_b &= 0 \\
\Gamma^a_{bc} A^b B_a + A^b \tilde{\Gamma}^a_{bc} B_a &= 0
\end{aligned} \tag{124}$$

where in the last line, we changed the dummy indices. Since (124) should be satisfied for arbitrary A^b and B_a, we conclude that

$$\tilde{\Gamma}^a_{bc} = -\Gamma^a_{bc} \tag{125}$$

Therefore, we can rewrite (122) as follows:

$$\nabla_c B_a = \partial_c B_a - \Gamma^b_{ac} B_b \tag{126}$$

By a similar argument, we can derive the following formula for the covariant derivative of higher rank tensors:

$$\nabla_c T^{a\cdots}_{b\cdots} = \partial_c T^{a\cdots}_{b\cdots} + \Gamma^a_{dc} T^{d\cdots}_{b\cdots} + \cdots - \Gamma^d_{bc} T^{a\cdots}_{d\cdots} - \cdots \tag{127}$$

15 Alternate derivation of the Christoffel symbol

The Christoffel symbol can be derived in another way. We can do this if we assume the following conditions:

$$\nabla_a g_{bc} = 0 \tag{128}$$

$$\Gamma^a_{bc} = \Gamma^a_{cb} \tag{129}$$

Let's see why the first condition must be satisfied. We have

$$\nabla_a V_b = \nabla_a (g_{bc} V^c) \tag{130}$$

However, the left-hand side is a tensor. Remembering that the indices in a tensor can be always lowered or raised by a metric, we can write

$$\nabla_a V_b = g_{bc} \nabla_a V^c \tag{131}$$

Combining (130) and (131), we have

$$\nabla_a (g_{bc} V^c) = g_{bc} \nabla_a V^c \tag{132}$$

However, by Leibniz rule, the left-hand side is equal to

$$(\nabla_a g_{bc}) V^c + g_{bc} \nabla_a V^c = g_{bc} \nabla_a V^c \tag{133}$$

Thus, we conclude (128). In other words, the metric is "sacred." Its covariant derivative is always zero. The second condition (129) is the torsion-free condition, which is reasonable to impose in light of equation (118).

Now, let's derive (119) in another way. From (127) and (128), we have:

$$\begin{aligned}
\nabla_d g_{bc} &= \partial_d g_{bc} - \Gamma^e_{db} g_{ec} - \Gamma^e_{dc} g_{be} = 0 \\
\nabla_b g_{cd} &= \partial_b g_{cd} - \Gamma^e_{bc} g_{ed} - \Gamma^e_{bd} g_{ce} = 0 \\
\nabla_c g_{db} &= \partial_c g_{db} - \Gamma^e_{cd} g_{eb} - \Gamma^e_{cb} g_{de} = 0
\end{aligned} \tag{134}$$

Subtracting the second and third equations from the first one yields the following:

$$\partial_d g_{bc} - \partial_b g_{cd} - \partial_c g_{db} + 2\Gamma^e_{bc} g_{ed} = 0 \tag{135}$$

where we have used (129), the torsion-free condition. Multiplying both sides of the above equation by g^{ad} yields equation (119), completing the proof.

Problem 11. Check this.

Problem 12. By showing the following, (Hint[3])

$$\Gamma^e_{ea} = \frac{1}{\sqrt{g}}\partial_a\sqrt{g} \tag{136}$$

show that

$$\nabla_e A^e = \frac{1}{\sqrt{g}}\partial_e(\sqrt{g}A^e) \tag{137}$$

Of course, when g is negative $\sqrt{g}$ should be replaced by $\sqrt{-g}$.

More generally, if $A^{e_1e_2\cdots e_n}$ is totally anti-symmetric in its indices, then we have

$$\nabla_{e_1} A^{e_1e_2\cdots e_n} = \frac{1}{\sqrt{g}}\partial_{e_1}(\sqrt{g}A^{e_1e_2\cdots e_n}) \tag{138}$$

if there is no torsion. You can show this by the symmetricity of the Christoffel symbol in the absence of torsion.

Problem 13. Find the Laplacian in the spherical coordinate system using (137), and check that it agrees with the one we found in our earlier article. (Hint[4])

Problem 14. In this problem, we will introduce an equivalent way of requiring that the covariant derivative of metric must vanish. Let's consider two vectors V^a and W^b. Their inner product is given by $g_{ab}V^aW^b$. Then, we will require that this inner product doesn't change when we parallel transport these two vectors in an *arbitrary* direction A^c. In other words,

$$\nabla_c V^a = 0, \qquad \nabla_c W^b = 0, \qquad A^c\nabla_c(g_{ab}V^aW^b) = 0. \tag{139}$$

As this should be true for any A^c, the last condition means

$$\nabla_c(g_{ab}V^aW^b) = 0 \tag{140}$$

Given this, show that $\nabla_c g_{ab} = 0$. (Hint[5])

16 The Riemann tensor

Except for the case of geodesics, all our parallel transports thus far have been along infinitesimal paths, taking the limit as ϵ approaches 0. A natural question to ask is: Does the parallel transport of a vector from a point A to another point B, depend on the path taken?

See Figure 4. Points A and C are on the equator of the earth, and point B lies on the north pole. Let's parallel transport a vector along two different paths from A to B. In the first path, the vector pointing north is transported to the north pole along the

[3] Use (395).
[4] $\nabla^2\phi = \nabla_e\nabla^e\phi$.
[5] Use Leibniz rule.

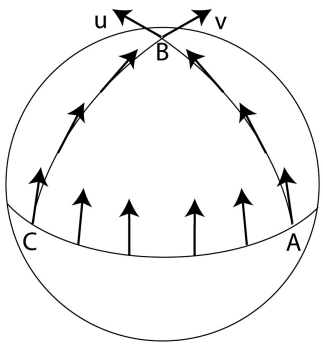

Figure 4: Parallel transporting a vector along two different paths on the earth

meridian. In the second path, the vector pointing north is transported along the equator from A to C, and then along the meridian from C to B. In both cases, the vector always pointed north, but the resulting two vectors are different at the north pole. Therefore, we see that parallel transport depends on the path.

One may equivalently regard this situation as parallel transporting a vector along a closed loop, rather than comparing the parallel transport along two different paths. For example, one can start on the north pole B with the given vector u and parallel transport this vector southward to A and parallel transporting this vector further along the equator until it hits C and parallel transporting this vector along the meridian up to the north pole which gives the vector v.

It is intuitively clear that when A and C are on the equator, the longer the distance between them, the bigger the difference between the two parallel transported vectors. In other words, this difference is proportional to the area encircled by the closed loop, the "triangle" ABC.

To calculate how much a vector changes as it is parallel transported around a loop, let us consider an infinitesimal version of such a case, as in Figure 5.

A vector V is parallel transported along two paths, path 1 and path 2. In the first case, the vector is parallel transported from the point x to $x + \epsilon_1 A$, then to $x + \epsilon_1 A + \epsilon_2 B$. The vector becomes $V_1(x + \epsilon_1 A + \epsilon_2 B)$. In the second case, the vector is parallel transported from the point x to $x + \epsilon_2 B$, then to $x + \epsilon_1 A + \epsilon_2 B$. The vector becomes $V_2(x + \epsilon_2 B + \epsilon_1 A)$.

We will take ϵ_1 and ϵ_2 to be infinitesimal. Then, we have:

$$V^a(x + \epsilon_1 A) - V^a(x) = \epsilon_1 A^c \nabla_c V^a(x) \tag{141}$$

Similarly, we have:

$$V_2^a(x + \epsilon_2 B + \epsilon_1 A) - V^a(x + \epsilon_2 B) = \epsilon_1 A^c \nabla_c V^a(x + \epsilon_2 B) \tag{142}$$

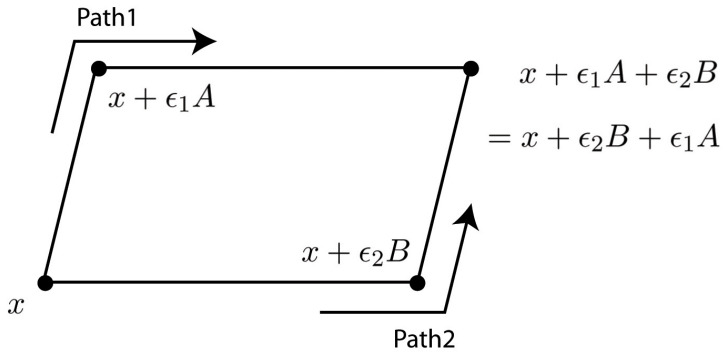

Figure 5: Parallel transporting a vector along two different paths

Now let's compare (142) with (141).

$$\begin{aligned}&(V_2^a(x+\epsilon_2 B+\epsilon_1 A)-V^a(x+\epsilon_2 B))-(V^a(x+\epsilon_1 A)-V^a(x))\\&=\epsilon_2 B^d\nabla_d(\epsilon_1 A^c\nabla_c V^a)=\epsilon_2 B^d\epsilon_1(\nabla_d A^c\nabla_c V^a+A^c\nabla_d\nabla_c V^a)\end{aligned}\tag{143}$$

Similar to (141), we have

$$V^a(x+\epsilon_2 B)-V^a(x)=\epsilon_2 B^c\nabla_c V^a(x)\tag{144}$$

Similarly, we have:

$$V_1^a(x+\epsilon_1 A+\epsilon_2 B)-V^a(x+\epsilon_1 A)=\epsilon_2 B^c\nabla_c V^a(x+\epsilon_1 A)\tag{145}$$

Its effect is simply exchange $\epsilon_1 A$ and $\epsilon_2 B$. By comparing (145) with (144), just as before, we obtain:

$$\begin{aligned}&(V_1^a(x+\epsilon_1 A+\epsilon_1 B)-V^a(x+\epsilon_1 A))-(V^a(x+\epsilon_2 B)-V^a(x))\\&=\epsilon_1 A^d\epsilon_2(\nabla_d B^c\nabla_c V^a+B^c\nabla_d\nabla_c V^a)=\epsilon_1 A^c\epsilon_2(\nabla_c B^d\nabla_d V^a+B^d\nabla_c\nabla_d V^a)\end{aligned}\tag{146}$$

Subtracting (143) from (146), we get:

$$\begin{aligned}&V_1^a(x+\epsilon_1 A+\epsilon_2 B)-V_2^a(x+\epsilon_2 B+\epsilon_1 A)\\&=\epsilon_1 A^c\epsilon_2 B^d(\nabla_c\nabla_d V^a-\nabla_d\nabla_c V^a)+\epsilon_1 A^c\epsilon_2(\nabla_c B^e\nabla_e V^a)-\epsilon_2 B^d\epsilon_1(\nabla_d A^e\nabla_e V^a)\end{aligned}\tag{147}$$

Given this, we are now going to use the fact that A and B are constant vectors. If they weren't Figure 5 would not be a parallelogram, as one side would be longer than the other. This fact implies:

$$\nabla_c B^e=\partial_c B^e+\Gamma^e_{dc}B^d=\Gamma^e_{dc}B^d\tag{148}$$

Similarly, we have:

$$\nabla_d A^e=\Gamma^e_{cd}A^c\tag{149}$$

Now we are ready to define the Riemann tensor $R^a{}_{bcd}$ as the amount a vector changes as it is parallel transported around a loop as follows:

$$R^a{}_{bcd}\epsilon_1 A^c \epsilon_2 B^d V^b = V_1^a(x + \epsilon_1 A + \epsilon_2 B) - V_2^a(x + \epsilon_2 B + \epsilon_1 A) \tag{150}$$

$$= \epsilon_1 A^c \epsilon_2 B^d (\nabla_c \nabla_d V^a - \nabla_d \nabla_c V^a + (\Gamma^e_{dc} - \Gamma^e_{cd})\nabla_e V^a) \tag{151}$$

In other words,

$$R^a{}_{bcd} V^b = \nabla_c \nabla_d V^a - \nabla_d \nabla_c V^a + (\Gamma^e_{dc} - \Gamma^e_{cd})\nabla_e V^a \tag{152}$$

Let us obtain the Riemann tensor in terms of the Christoffel symbols. Noting that $\nabla_d V^a$ and $\nabla_c V^a$ are (1,1) tensors, we can apply definition (127) to obtain:

$$\nabla_c \nabla_d V^a = \partial_c(\partial_d V^a + \Gamma^a_{bd} V^b) + \Gamma^a_{ec}(\partial_d V^e + \Gamma^e_{bd} V^b) - \Gamma^e_{dc}\nabla_e V^a \tag{153}$$

$$\nabla_d \nabla_c V^a = \partial_d(\partial_c V^a + \Gamma^a_{bc} V^b) + \Gamma^a_{ed}(\partial_c V^e + \Gamma^e_{bc} V^b) - \Gamma^e_{cd}\nabla_e V^a \tag{154}$$

Plugging (153) and (154) to (152), we get what we want as follows:

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc} \tag{155}$$

Notice also that in the torsion-free case, (152) becomes:

$$R^a{}_{bcd} V^b = \nabla_c \nabla_d V^a - \nabla_d \nabla_c V^a \tag{156}$$

Note that the Riemann tensor is indeed a tensor, since the right-hand side of (156) is a tensor. Another remarkable fact is that $R^a{}_{bcd}$ is independent of V, as you can see from (155). Naively, it may seem that the right-hand side of (156) depends on how V is defined in the neighborhood of a given point, since the covariant derivatives compare the values of V around the neighborhood of that point. However, the right-hand side of (156) shows that this value only depends on how V is defined on the given point. Indeed, the Riemann tensor is a function of Christoffel symbols only, which themselves are functions of the metric and the torsion only; the Riemann tensor is truly an inherent property of spacetime rather than a property of a vector field.

Before concluding this section, I want to note that the "loop" in "loop quantum gravity," a theory of quantum gravity, refers to loops such as the one formed by Path 1 and Path 2 in Figure 5. In place of the metric, such loops play a fundamental role in loop quantum gravity.

17 Properties of the Riemann tensor

In this section, we present various identities which the Riemann tensor satisfies. First of all, it is easy to see that the Riemann tensor is antisymmetric in its first two indices:

$$\begin{aligned} \nabla_d \nabla_c V^a - \nabla_c \nabla_d V^a &= R^a{}_{bdc} V^b \\ = -(\nabla_c \nabla_d - \nabla_d \nabla_c) V^a &= -R^a{}_{bcd} V^b \end{aligned} \tag{157}$$

and thus

$$R^a{}_{bdc} = -R^a{}_{bcd} \tag{158}$$

Equivalently, by lowering the indices:

$$R_{abdc} = -R_{abcd} \tag{159}$$

Another identity that we can derive is the following:

$$R_{bacd} = -R_{abcd} \tag{160}$$

To see this, first observe the following, using (127):

$$(\nabla_c \nabla_d - \nabla_d \nabla_c) T^{a\cdots}_{b\cdots} = R^a{}_{ecd} T^{e\cdots}_{b\cdots} + \cdots - R^e{}_{bcd} T^{a\cdots}_{e\cdots} - \cdots$$

Applying this to a covector V_a yields:

$$(\nabla_c \nabla_d - \nabla_d \nabla_c) V_a = -R^b{}_{acd} V_b = -R_{bacd} V^b \tag{161}$$

$$= (\nabla_c \nabla_d - \nabla_d \nabla_c) V^e g_{ea} = R^e{}_{bcd} V^b g_{ea} \tag{162}$$

$$= R_{abcd} V^b \tag{163}$$

By comparing (161) and (163), we get (160).

(159) and (160) show that the first two and last two indices of the Riemann tensor are antisymmetric.

Note the following identity, which can be derived by explicit calculation, using the symmetry of the Christoffel symbol:

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) V_c + (\nabla_b \nabla_c - \nabla_c \nabla_b) V_a + (\nabla_c \nabla_a - \nabla_a \nabla_c) V_b = 0 \tag{164}$$

For those readers familiar with differential forms, this equation is equivalent to $d^2 V = 0$. Therefore, we obtain:

$$
\begin{aligned}
(\nabla_a\nabla_b - \nabla_b\nabla_a)V_c + (\nabla_b\nabla_c - \nabla_c\nabla_b)V_a + (\nabla_c\nabla_a - \nabla_a\nabla_c)V_b &= 0 \\
-R_{dcab}V^d - R_{dabc}V^d - R_{dbca}V^d &= 0 \\
R_{dcab} + R_{dabc} + R_{dbca} &= 0 \qquad (165)
\end{aligned}
$$

Using the identities (159), (160) and (165), we obtain:

$$
R_{abcd} = R_{cdab} \qquad (166)
$$

Problem 15. Show this! (Hint[6])

Problem 16. Show $R_{abcd}g^{ab} = R_{abcd}g^{cd} = 0$.

Finally, we derive the so-called Bianchi identities, given by the following formula:

$$
\nabla_a R^e{}_{dbc} + \nabla_b R^e{}_{dca} + \nabla_c R^e{}_{dab} = 0 \qquad (167)
$$

To derive this, observe the following:

$$
\begin{aligned}
(\nabla_a\nabla_b - \nabla_b\nabla_a)\nabla_c V^e &= R^e{}_{dab}\nabla_c V^d - R^f{}_{cab}\nabla_f V^e \qquad (168) \\
\nabla_a(\nabla_b\nabla_c V^e - \nabla_c\nabla_b V^e) &= \nabla_a(R^e{}_{dbc}V^d) = V^d\nabla_a R^e{}_{dbc} + R^e{}_{dbc}\nabla_a V^d \qquad (169)
\end{aligned}
$$

From (168), we obtain:

$$
\begin{aligned}
&(\nabla_a\nabla_b - \nabla_b\nabla_a)\nabla_c V^e + (\nabla_b\nabla_c - \nabla_c\nabla_b)\nabla_a V^e + (\nabla_c\nabla_a - \nabla_a\nabla_c)\nabla_b V^e \qquad (170) \\
&= R^e{}_{dab}\nabla_c V^d + R^e{}_{dbc}\nabla_a V^d + R^e{}_{dca}\nabla_b V^d - (R^f{}_{cab} + R^f{}_{abc} + R^f{}_{bca})\nabla_f V^e \\
&= R^e{}_{dab}\nabla_c V^d + R^e{}_{dbc}\nabla_a V^d + R^e{}_{dca}\nabla_b V^d \qquad (171)
\end{aligned}
$$

where in the last equation we used (165). From (169), we obtain:

$$
\begin{aligned}
&\nabla_a(\nabla_b\nabla_c V^e - \nabla_c\nabla_b V^e) + \nabla_b(\nabla_c\nabla_a V^e - \nabla_a\nabla_c V^e) + \nabla_c(\nabla_a\nabla_b V^e - \nabla_b\nabla_a V^e) \qquad (172) \\
&= V^d(\nabla_a R^e{}_{dbc} + \nabla_b R^e{}_{dca} + \nabla_c R^e{}_{dab}) + R^e{}_{dbc}\nabla_a V^d + R^e{}_{dca}\nabla_b V^d + R^e{}_{dab}\nabla_c V^d \qquad (173)
\end{aligned}
$$

Noticing that (170) is equal to (172), we can conclude that (171) should be equal to (173), which implies (167).

18 The Ricci tensor, the Ricci scalar, and the Einstein tensor

The Ricci tensor is defined by the following formula:

$$
R_{ab} = R^c{}_{acb} = R_{dacb}g^{dc} \qquad (174)
$$

[6] The answer is in Appendix F.

Identity (166) and the symmetricity of the metric tensor implies that the Ricci tensor is symmetric:

$$R_{ab} = R_{ba} \tag{175}$$

The Ricci scalar, or the scalar curvature, R is defined by the following formula:

$$R = g^{ab} R_{ab} \tag{176}$$

The Einstein tensor G_{ab} is defined by:

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R \tag{177}$$

The Einstein tensor satisfies the contracted Bianchi identity:

$$\nabla^a G_{ab} = 0 \tag{178}$$

which can be derived from the Bianchi identity (167) as follows:

$$\begin{aligned}
\nabla_a R_{bcd}{}^a + \nabla_b R_{cad}{}^a + \nabla_c R_{abd}{}^a &= 0 \\
\nabla_a R_{bcd}{}^a + \nabla_b R_{cd} - \nabla_c R_{bd} &= 0 \\
g^{bd}(\nabla_a R_{bcd}{}^a + \nabla_b R_{cd} - \nabla_c R_{bd}) &= 0 \\
\nabla_a R_c{}^a + \nabla_b R_c{}^b - \nabla_c R &= 0 \\
2\nabla_a R_c{}^a - \nabla_c R &= 0 \\
2\nabla_a (R_c{}^a - \delta_c^a \frac{R}{2}) &= 0 \\
\nabla_a G_c{}^a &= 0
\end{aligned} \tag{179}$$

19 Einstein-Hilbert action

Now, we have provided all the materials necessary to understand the brief preview of general relativity presented in Section 9. In this section, we justify the form of the Lagrangian and present a more rigorous derivation of Einstein's equation.

As argued in Section 9, the action must contain an integrand proportional to the volume element, given by:

$$d^4x\sqrt{-g} \tag{180}$$

The integral of the Lagrangian density multiplied by this volume factor gives the action. What should the Lagrangian density associated to the curvature be? Lagrangian density is a scalar, so it must have either no indices, or only contracted indices. By contracted indices, I mean the repeated indices found in the Einstein summation convention. Since the Lagrangian density must be covariant, it has to be formed from the Ricci scalar, the Ricci tensor, and the Riemann tensor, which are themselves covariant.

Therefore, we may write the Lagrangian density in the following general form:

$$\mathcal{L}_g = c_0 + c_1 R + c_2 R^2 + \cdots + d_1 R_{ab} R^{ab} + d_2 R_{ab} R^{ab} R + \cdots + e_1 R_{abcd} R^{abcd} + \cdots \quad (181)$$

where the c's, d's, and e's are constants, and $\mathcal{L}_g$ denotes the Lagrangian density associated to gravitation. However, for small R all terms but the first two become negligible. Setting $c_0 = -2\Lambda/(16\pi G)$ and $c_1 = 1/(16\pi G)$, we obtain:

$$\mathcal{L}_g = \frac{R - 2\Lambda}{16\pi G} \quad (182)$$

which reproduces the sum of (80) and (81). Now, recall the full action mentioned in Section 9:

$$S = \int d^4x \sqrt{-g} \left(\frac{R - 2\Lambda}{16\pi G} + \mathcal{L}_m \right) \quad (183)$$

where $\mathcal{L}_m$ is the matter Lagrangian density. The first term of this action is called the Einstein-Hilbert action. Let's consider the variation of the full action (183), which should be zero by the action principle:

$$0 = \delta S = \int d^4x \frac{\delta g^{ab}(\sqrt{-g}(R_{ab} - 8\pi G T_{ab})) + \sqrt{-g} g^{ab} \delta R_{ab} + \delta\sqrt{-g}(R - 2\lambda)}{16\pi G} \quad (184)$$

where we have used the following definition of the energy-momentum tensor T_{ab}:

$$T_{ab} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{ab}} \quad (185)$$

In Section 21, we will talk more about the energy-momentum tensor.

A closer inspection reveals that the second term of (184) is a total derivative, which doesn't affect the Euler-Lagrange equation (by Stokes' Theorem). See Appendix E. Therefore,

$$\begin{aligned} 0 &= \int d^4x [\delta g^{ab}(\sqrt{-g}(R_{ab} - 8\pi G T_{ab})) + \delta\sqrt{-g}(R - 2\Lambda)] && (186) \\ &= \int d^4x [\delta g^{ab}(\sqrt{-g}(R_{ab} - 8\pi G T_{ab})) + \delta g^{ab}(-\frac{1}{2}\sqrt{-g} g_{ab}(R - 2\Lambda))] && (187) \end{aligned}$$

Therefore, we recover Einstein's equation (93). When the cosmological constant Λ is absent, we can write:

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab} \quad (188)$$

Using the Einstein tensor (177), this becomes

$$G_{ab} = 8\pi G T_{ab} \quad (189)$$

In the limit where g_{ab} approaches the flat metric, the solutions of the above equation can be reduced to Newtonian gravity, as we will see later in the article.

Let us conclude this section with some remarks. Einstein didn't initially derive Einstein's equation by following the steps presented in this section. He first obtained Einstein's equation by another method and then succeeded in finding out the action that derives Einstein's equation. However, the action so obtained was not as simple as the Einstein-Hilbert action but an ugly, non-covariant combination of Christoffel symbols. The Einstein-Hilbert action was first found by Hilbert in 1915. Considering also the fact that Newton used a geometric method rather than an algebraic one to derive Kepler's laws and the fact that Heisenberg came up with quantum mechanics in a roundabout way, it is often the case that it is better to study physics from modern reviews rather than from the original papers even though it is sometimes interesting to find out how great physicists came up with their ideas for the first time. Nevertheless, this you can do after learning things from a modern perspective first.

20 Energy-Momentum Conservation and revisiting the contracted Bianchi identity

In this section, we will derive the equation for energy-momentum conservation and reinterpret the contracted Bianchi identity (178).

Consider the action due to matter as follows:

$$S_m = \int d^4x \sqrt{-g} \mathcal{L}_m \tag{190}$$

This is the third term of (183). Notice that the above term must remain invariant under a change of coordinates, since the matter action cannot depend on the coordinate one chooses; this is a property of integration, which has nothing to do with action principle.

Building on this observation, consider the following change of coordinates:

$$x'^a = x^a + \epsilon \xi^a \tag{191}$$

where ϵ is infinitesimally small. Under this infinitesimal change of coordinates, how does the metric change? We can plug the above equation into (20) to get:

$$\begin{aligned} g_{ab}(x) &= g'_{ij}(x+\epsilon\xi)(\delta^i_a + \epsilon\partial_a\xi^i)(\delta^j_b + \epsilon\partial_b\xi^j) \\ g_{ab}(x) &= (g'_{ij}(x) + \epsilon\xi^e\partial_e g'_{ij})(\delta^i_a + \epsilon\partial_a\xi^i)(\delta^j_b + \epsilon\partial_b\xi^j) \\ g_{ab}(x) &= g'_{ab}(x) + \epsilon(\xi^e\partial_e g'_{ab} + g'_{ad}\partial_b\xi^d + g'_{bd}\partial_a\xi^d) \end{aligned} \tag{192}$$

to first order in ϵ. We can rewrite this as follows:

$$g'_{ab}(x) = g_{ab}(x) - \epsilon(\xi^e\partial_e g'_{ab} + g'_{ad}\partial_b\xi^d + g'_{bd}\partial_a\xi^d) \tag{193}$$

If we denote the variation of the metric by δg_{ab} and use (126) and (135), we get:

$$\begin{aligned}\delta g_{ab} &= -\epsilon(\xi^e \partial_e g'_{ab} + g'_{ad}\partial_b \xi^d + g'_{bd}\partial_a \xi^d) \\ &= -\epsilon(\nabla_a \xi_b + \nabla_b \xi_a) \end{aligned} \tag{194}$$

Now we are ready. Under (191), the change of the matter action must be zero. In other words,

$$\begin{aligned} 0 &= \delta S_m = -\int d^4x \delta g_{ab} T^{ab} \\ &= \epsilon \int d^4x (\nabla_a \xi_b + \nabla_b \xi_a) T^{ab}) \\ &= 2\epsilon \int d^4x (\nabla_a \xi_b) T^{ab} \end{aligned} \tag{195}$$

where in the last line we used the change of dummy indices and the symmetricity of the energy-momentum tensor. Going one step further, we can reexpress the above equation as:

$$0 = 2\epsilon \int d^4x \left(\nabla_a (\xi_b T^{ab}) - \xi_b (\nabla_a T^{ab}) \right) \tag{196}$$

Since the first term is a total derivative, or a so-called surface term, it vanishes as long as we justifiably assume T^{ab} vanishes at infinity. Therefore, the second term must vanish. As this must happen, for any arbitrary ξ_b, we conclude:

$$\nabla_a T^{ab} = 0 \tag{197}$$

This is the energy-momentum conservation equation, derived as a consequence of the invariance of the theory under change of coordinates. In the next section, we will see how this corresponds to a more familiar version of the energy-momentum conservation equation.

Similarly, by considering the following Einstein-Hilbert action instead of (190),

$$S_{EH} = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G} \right) \tag{198}$$

and imposing the condition that the total action is invariant under coordinate transformation, one can derive the contracted Bianchi identity, which we reproduce here for convenience:

$$\nabla_a G^{ab} = 0 \tag{199}$$

We should be relieved that the above equation is satisfied. Otherwise, it would mean that the total action due to the curvature of spacetime (i.e the Einstein-Hilbert action)

is not invariant under the coordinate transformation, but depends on the coordinate one chooses. In such a hypothetical case, Einstein's equation, which is derived from the Einstein-Hilbert action, would be satisfied only for certain coordinates but not for others. Therefore, we can regard the contracted Bianchi identity as an equation that must be satisfied if Einstein's equation is correct for any coordinates one chooses; it is needed for consistency.

As an aside, there are consistency equations that must be satisfied in string theory as well; what are called "anomalies" have to be zero when calculated, as much as the right-hand side of (199) must be zero. There are three types of anomalies in string theory: gauge anomaly, gravitational anomaly and mixed anomaly. Gauge anomaly must be zero in order that what is called "gauge symmetry" is respected. We will talk about the gauge symmetry in the next article, even though we will not talk about gauge anomaly there. We will talk about it in our later article on quantum field theory. Gravitational anomaly must be zero in order that the theory is invariant under the change of coordinate as we considered in this section. This invariance is often called "general covariance" or "diffeomorphism invariance." Roughly speaking diffeomorphism is a differentiable map (i.e. a function) from a manifold to another manifold. In the case in which "another manifold" is the original manifold itself, diffeomorphism can be regarded as coordinate transformation. Mixed anomaly is a kind of mixture of gauge anomaly and gravitational anomaly. We will not talk more about it other than that as it is very technical. In 1984, Green and Schwarz showed that the three types of anomalies are cancelled (i.e. equal to zero) in certain types of string theory. This is called "first superstring revolution." String theory became mainstream.

Let me conclude this section with another historical remark. On November 11th 1915, Einstein guessed the following wrong field equation:

$$R_{ab} = 8\pi G T_{ab} \tag{200}$$

Even though he knew the equation for energy momentum conservation, he did not yet know the Bianchi identities then. Therefore, he did not immediately realize that the above equation was wrong. (Take the covariant derivative of both sides. The left hand-side is not zero, while the right hand-side is zero.) Anyhow, he came up with the correct equation on November 25th which is the following:

$$R^{ab} = 8\pi G(T^{ab} - \frac{1}{2}g^{ab}T^{cd}g_{cd}) \tag{201}$$

The above equation is equivalent to Einstein's equation (188).

Problem 17. Prove this. (Hint[7])

[7]The solution is in Section 22.

21 Energy-Momentum tensor

In this section, we will obtain the energy-momentum tensor for what is usually called "dust" and, as advertised, show that it is consistent with (197), our earlier equation for energy-momentum conservation.

Recall that we have actually defined the energy-momentum tensor in (185). However, we will not use this formula to derive the energy-momentum tensor for dust, as I do not know how to do it. Nevertheless, (185) is still useful in many other cases. One such example is the electromagnetic field, since its Lagrangian is explicitly known.

Now, let's think about how T^{ab} should be given. Since this is a tensor, there should be covariance. So, we have to find a symmetric rank-2 tensor. There are two ways to obtain one. One would be simply finding a symmetric rank-2 tensor, and the other one would be taking the tensor-product of a rank-1 tensor with itself. Then, a generic symmetric rank-2 tensor would be simply a sum of these two forms. One example of symmetric rank-2 tensor is the metric, g^{ab}. One example of a rank-1 tensor is the 4-velocity, given as follows:

$$u^a = \frac{dx^a}{d\tau} \tag{202}$$

where τ is the proper time. It is easy to see that the 4-velocity is given by:

$$u^a = \gamma(1, u_x, u_y, u_z) \tag{203}$$

where $u_x = dx/dt$, and so on.

Then we can write the energy momentum tensor as follows:

$$T^{ab} = Au^a u^b + Bg^{ab} \tag{204}$$

In case of dust, A turns out to be ρ_0 the proper density and B turns out to be 0. In other words,

$$T^{ab} = \rho_0 u^a u^b \tag{205}$$

The proper density is the mass density (or equivalently the energy density) measured by an observer moving with the dust. Now using (203), we get

$$T^{00} = \gamma^2 \rho_0 \tag{206}$$

This expression has an obvious physical interpretation. As the dust moves, its energy increases by a factor γ, and its length decreases by a factor γ. So, the energy density increases by a factor γ^2. Therefore, T^{00} is the energy density in which relativistic effects are considered. So, let's call the energy density T^{00} "ρ," and consider the case in which

we live in a flat, Minkowski, Cartesian coordinate system. Then, the energy momentum conservation law (197) is simply the following:

$$\partial_b T^{ab} = 0 \tag{207}$$

When $a = 0$ this reduces to:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial_x}(\rho u_y) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z) = 0 \tag{208}$$

which is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{209}$$

This is exactly the equation of continuity! So, the mass of dust is conserved which implies the conservation of energy. In the case $a = 1, 2, 3$, we obtain the Navier-Stokes equations, which play a pivotal role in fluid mechanics. Here, we will not explain it in more detail.

Anyhow, let's obtain the form for the energy momentum tensor for what is called "perfect fluid." Perfect fluid is completely characterized by its mass density ρ and its pressure p. Let's consider the energy-momentum tensor in a flat space when the observer is not moving with respect to the perfect fluid. Then, we have

$$u^a = (1, 0, 0, 0) \tag{210}$$

Plugging this to (204), we have

$$T^{ab} = \begin{pmatrix} A - B & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & B \end{pmatrix} \tag{211}$$

We know that T^{00} is the energy density which is mass density. Of course, this is given by ρ. Also, considering the Navier-Stokes equations, it turns out that $T^{11} = T^{22} = T^{33} = p$. Thus, we have

$$A - B = \rho, \qquad B = p \tag{212}$$

Thus, for a perfect fluid, the energy-momentum tensor is given by:

$$T^{ab} = (\rho_0 + p)u^a u^b + p g^{ab} \tag{213}$$

This formula plays an important role in cosmology. Dust is an object, whose pressure is negligible compared to its mass density (i.e. energy density). For example, the air on the Earth can be considered as dust, as the kinetic energy of the air molecules are much less than their mass multiplied by c^2 (i.e. the rest mass energy). (Remember from our

earlier article "Kinetic theory of gas" that pressure is in the order of kinetic energy per volume.)

Problem 18. In our later article, we will see that the Lagrangian density for electromagnetic field will be given as follows:

$$\mathcal{L}_m = -\frac{1}{4}F^{ab}F_{ab} \tag{214}$$

Use (185) to obtain the energy-momentum tensor for electromagnetic field. (Hint[8]) Check in particular that the energy density of the electromagnetic field agrees with the well-known result in flat space:

$$T_{00} = \frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2 \tag{215}$$

Also, obtain expressions for T_{01}, T_{02}, T_{03} and check that they coincide with Poynting vector. Notice that this is expected since (207) implies

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{S} = 0 \tag{216}$$

which is exactly the equation in "Poynting vector" if there is no electric charge.

22 Newtonian limit

If general relativity is a correct theory, its equations should reduce to those of Newtonian gravity in a suitable limit. Showing this is the aim of this section.

First, let's consider Newtonian gravity. We have:

$$\frac{d^2x^i}{dt^2} = -\nabla\Phi = -\partial^i\Phi \tag{217}$$

where i refers to the spatial index (i.e. 1 to 3) and Φ is the gravitational potential which satisfies the Poisson equation:

$$\nabla^2\Phi = 4\pi G\rho \tag{218}$$

where ρ is the density.

Then, the Newtonian equation of motion (217) should be derived from the geodesic equation (118), since an object always moves along the geodesics according to general relativity.

Now, notice that the Newtonian limit is the case in which c, the speed of light is considered very large. In this limit it follows that:

$$\frac{dx^0}{d\tau} = \frac{cdt}{d\tau} \gg \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau} \tag{219}$$

$$d\tau^2 = c^2dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \approx c^2dt^2 \tag{220}$$

[8] $-\frac{1}{4}F^{ab}F_{ab} = -\frac{1}{4}g^{ac}g^{bd}F_{cd}F_{ab}$

$$d\tau \approx cdt \tag{221}$$

$$\partial_0 = \frac{\partial}{c\partial t} \ll \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \tag{222}$$

Then, for i=1 to 3, we have:

$$\begin{aligned}
0 &= \frac{d^2x^i}{d\tau^2} + \Gamma^i_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} && (223)\\
&= \frac{d^2x^i}{d\tau^2} + \Gamma^i_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} && (224)\\
&= \frac{d^2x^i}{dt^2} + \Gamma^i_{00} && (225)\\
&= \frac{d^2x^i}{dt^2} + \frac{1}{2}g^{ij}(\partial_0 g_{j0} + \partial_0 g_{0j} - \partial_j g_{00}) && (226)\\
&= \frac{d^2x^i}{dt^2} - \frac{1}{2}g^{ij}\partial_j g_{00} && (227)\\
&= \frac{d^2x^i}{dt^2} - \frac{1}{2}\partial^i g_{00} && (228)
\end{aligned}$$

where we used (219) from the first line to the second line, and we used (221) from the second line to the third line, and we used (222) from the fourth line to the fifth line and from the fifth line to the sixth line.

Consider also that in the Newtonian limit the metric should deviate only slightly from flat-Minkowskian metric, which is given by:

$$\eta_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{229}$$

Comparing (217) and (228), we should have:

$$g_{00} = -(1 + 2\Phi) \tag{230}$$

In light of (218), we need to derive

$$\nabla^2 g_{00} = -8\pi G\rho \tag{231}$$

from Einstein's equation $G_{ab} = 8\pi G T_{ab}$. In fact the factor $8\pi G$ in Einstein's equation is to match the $8\pi G$ in the above equation. We will see this shortly.

Einstein's equation says:

$$\begin{aligned}
R_{ab} - \frac{1}{2}Rg_{ab} &= 8\pi G T_{ab} && (232)\\
g^{ab}(R_{ab} - \frac{1}{2}Rg_{ab}) &= 8\pi G T && (233)\\
-R &= 8\pi G T && (234)
\end{aligned}$$

Now plugging this into (232) yields:

$$R_{ab} = 8\pi G(T_{ab} - \frac{1}{2}Tg_{ab}) \tag{235}$$

In the Newtonian limit, matter moves very slowly compared to the speed of light. Therefore, in this limit, in light of (203) and (205), the energy momentum tensor takes the value $T_{00} = \rho$ and 0 for all other components. Therefore, we have:

$$T = g^{00}T_{00} = -\rho \tag{236}$$

and from (235), we have:

$$R_{00} = 8\pi G(T_{00} - \frac{1}{2}Tg_{00}) = 4\pi\rho \tag{237}$$

To express this quantity in terms of the metric, we have:

$$\begin{aligned} R_{00} &= R^i{}_{0i0} = \partial_i\Gamma^i_{00} - \partial_0\Gamma^i_{i0} + \Gamma^i_{i\lambda}\Gamma^\lambda_{00} - \Gamma^i_{0\lambda}\Gamma^\lambda_{i0} && (238) \\ &\approx \partial_i\Gamma^i_{00} && (239) \end{aligned}$$

where from the first line to the second line we used (222), and the fact that Γ is first order in the perturbation to the flat Minkowskian metric so the terms quadratic in Γ can be ignored. Using Γ^i_{00} already calculated in (225) and (227), we have:

$$4\pi\rho = R_{00} = -\frac{1}{2}g^{ij}\partial_i\partial_j g_{00} = -\frac{1}{2}\nabla^2 g_{00} \tag{240}$$

So, we recover (231) as advertised. This completes the proof that Einstein's equation reduces to Newtonian gravity in a suitable limit.

23 Schwarzschild black hole

A little more than a month after the publication of Einstein's paper on general relativity, Schwarzschild found the following solution for Einstein's equation.

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{241}$$

In other words, the above metric, called "Schwarzschild metric" satisfies $G_{ab} = 0$, except when $r = 0$ for which G_{ab} is singular.

Let's compare this with the Newtonian limit considered in the last section. If a point mass M is located at $r = 0$, we have:

$$\Phi = -\frac{GM}{r} \tag{242}$$

$$g_{00} = -(1+2\Phi) = -\left(1 - \frac{2GM}{r}\right) \tag{243}$$

So, the above metric corresponds to the system in which an object with mass $M = r_s/2$ is placed at the origin, if we use the convention $G = c = 1$ as common in the literature. If you don't use this convention, $r_s = 2M$ is replaced by $r_s = \frac{2GM}{c^2}$. Notice also that when r goes to infinity, the Schwarzschild metric reduces to the flat Minkowski metric in spherical coordinates. This is expected as the gravity is very weak and negligible in such a limit. In other words, (241) can be re-written as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{244}$$

Let's closely look at this formula. When r is smaller than $2M$, g_{tt} becomes positive while g_{rr} becomes negative. So, a really strange thing must happen in the region $r < 2M$. The roles of space and time are exchanged. Actually, it is called black hole. The region $r < 2M$ is inside the black hole, the region $r > 2M$ is outside the black hole and the region $r = 2M$ is called the black hole horizon. The black hole given by (244) is called the Schwarschild black hole.

Now that we have briefly discussed the inside of the black hole, let's see what happens outside the black hole. To this end, let's compare the time of two clocks, which are at rest at positions $r = r_1$ and $r = r_2$. The proper time $\Delta\tau$ is given by $\Delta\tau^2 = |g_{tt}|\Delta t^2$ which implies:

$$\Delta\tau_1 = \sqrt{|g_{tt}(r = r_1)|}\Delta t \tag{245}$$

$$\Delta\tau_2 = \sqrt{|g_{tt}(r = r_2)|}\Delta t \tag{246}$$

Therefore, we see that the time of clocks flow at the ratio:

$$\frac{\Delta\tau_1}{\Delta\tau_2} = \sqrt{\frac{g_{tt}(r = r_1)}{g_{tt}(r = r_2)}} \tag{247}$$

Considering the actual value of g_{tt} in (244), we conclude that time flows at a slower rate at a place closer to the black hole. In particular, time flows almost infinitely slowly near the horizon. We say that time is infinitely "dilated" around the horizon.

Another way of seeing that time should flow more slowly at a place closer to the black hole is the following. Suppose you send a light signal with frequency f_1 from the point $(r = r_1)$ to $(r = r_2 > r_1)$. Since light has energy hf_1, it's "mass" should be hf_1/c^2 from the famous formula $E = Mc^2$. Therefore, the light loses energy as it goes from a lower place r_1 to a higher place r_2. The energy change is approximately given by:

$$\Delta E \approx GM\frac{hf_1}{c^2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \tag{248}$$

which should be equal to $h(f_2 - f_1)$ So, we conclude:

$$\frac{f_2 - f_1}{f_1} \approx -\frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{249}$$

Since the frequency is the inverse of the period, (249) implies:

$$\frac{\tau_2 - \tau_1}{\tau_1} \approx \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{250}$$

We can easily see that this coincides with (247). In other words, we checked that $r_s = \frac{2GM}{c^2}$ makes sense, in another way.

Now, let's check the well-known statement that even light, the fastest thing, cannot escape the black hole. To this end, let's calculate the time that light emitted at $r = r_i > 2M$ (i.e. outside of the black hole) reaches us at $r = r_f > 2M$. For simplicity, we will assume that the light moves along the radial direction. In other words, θ and ϕ are fixed, which means $d\theta = d\phi = 0$. Consider also the fact that the proper time of light is always zero. Thus, from (244), we have:

$$\begin{aligned} 0 &= -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \\ \int dt &= \int_{r_i}^{r_f} \frac{dr}{\left(1 - \frac{2M}{r}\right)} \\ t &= r_f - r_i + 2m \ln \frac{r_f - 2M}{r_i - 2M} \end{aligned} \tag{251}$$

Notice that t diverges as r_i approaches $2M$. If you carefully think about its origin, it's because g_{tt} approaches zero near the horizon. In other words, time is infinitely dilated around the horizon, as stated above. The divergence of t also implies that it takes a huge amount of time for light to travel from just outside of the black hole horizon to us. And, it also means that it takes an infinite amount of time for light to travel from the black hole horizon to us, wherever we are, as long as we are outside of black hole. Going further, we can therefore say that even light cannot escape from the inside of the black hole. This is true for any black hole, not just a Schwarzschild black hole.

The Schwarzschild black hole is not the only black hole known. The Schwarzschild black hole has spherical symmetry which implies that it is not rotating. It also has no electric charge. A Kerr black hole, described by the Kerr metric, is rotating but has no electric charge. A Reissner-Nordström black hole, described by the Reissner-Nordström metric, is not rotating but has electric charge. A Kerr-Newman black hole, described by the Kerr-Newman metric, is rotating and has electric charge. So, the Schwarzschild metric is a special case of the other three metrics.

24 Experimental tests of general relativity

As advertised, in this section we will examine the two most prominent experimental tests of general relativity, the ones first calculated by Albert Einstein.

We consider the motion of Mercury first. To obtain the motion of Mercury, we will use the Schwarzschild metric (244), even though it is a solution for a black hole, as it is also the solution to Einstein's equation when only one object is present, namely, the Sun at the origin. (We don't count Mercury, as Mercury moves according to the metric created by the presence of the Sun.) Of course, in this case, the black hole is not present as r_s, called the "Schwarzschild radius," is much smaller than the radius of the Sun. In other words, as long as Mercury moves outside the Sun, the metric due to the Sun in this region is duly given by the Schwarzschild metric.

From the symmetry of the system and Newtonian intuition, it is clear that Mercury will move around a plane. (One can actually justify this observation by solving the complicated equation of motion for Mercury without using this observation, but we will not do so.) Considering this, we can carefully choose the coordinates in such a way that $\theta = \pi/2$, which in turn implies $d\theta = 0$ as θ is a constant. From (244) this choice renders the following:

$$\left(1 - \frac{2M}{r}\right)\dot{t}^2 - \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 1 \tag{252}$$

where the dots denote the differentiation with respect to τ, the proper time, and we used the relation between the proper time and the proper distance, i.e. $d\tau^2 = -ds^2$. Now, to calculate the geodesic, which would be the path of Mercury, we can use (112), of which the left-hand side is given by (105) in case $u = \tau$. We could directly use (112), but using (105) for $u = \tau$ turns out to be much simpler, as it allows us to easily obtain $r(\phi)$ in terms of conserved quantities, without getting bogged down in explicitly inverting $\phi(\tau)$ and plugging this back into $r(\tau)$. Also, it turns out that for (105) we need to consider the cases $x^a = t$ and $x^a = \phi$ only, excluding $x^a = r$, as these equations will yield $t = t(\tau)$ and $\phi = \phi(\tau)$, which we can, then, plug into (252) to obtain $r = r(\tau)$. In any case, $x^a = t$ and $x^a = \phi$ yield:

$$\frac{d}{d\tau}\left[\left(1 - \frac{2M}{r}\right)\dot{t}\right] = 0 \tag{253}$$

$$\frac{d}{d\tau}(r^2\dot{\phi}) = 0 \tag{254}$$

which imply

$$\left(1 - \frac{2M}{r}\right)\dot{t} = k \tag{255}$$

$$r^2\dot{\phi} = l \tag{256}$$

for some constants k and l. Plugging k into (252), we get:

$$\frac{k^2}{1-2M/r} - \frac{\dot{r}^2}{1-2M/r} - r^2\dot{\phi}^2 = 1 \tag{257}$$

Actually k corresponds to relativistic energy.

Problem 19. Invert the above expression to obtain k in terms of other variables and their derivatives. Then, show that it indeed reduces to ordinary energy (i.e. rest energy + kinetic energy + potential energy) in the Newtonian limit.

If we set $u = r^{-1}$, we have (**Problem 20.**):

$$\dot{r} = -l\frac{du}{d\phi} \tag{258}$$

Plugging this into (257), we get:

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{k^2-1}{l^2} + \frac{2M}{l^2}u + 2mu^3 \tag{259}$$

Differentiating this with respect to ϕ, we get:

$$\frac{d^2u}{d\phi^2} = -u + \frac{M}{l^2} + 3Mu^2 \tag{260}$$

(**Problem 21.** Show this.)

This equation has the same structure as a harmonic oscillator problem if we linearize the right-hand side around the equilibrium point. The equilibrium point u_0 can be found when the right-hand side is zero. We get:

$$u_0 = \frac{1-\sqrt{1-12M^2/l^2}}{6M} \tag{261}$$

Now, if we differentiate the right-hand side and plug in u_0 to linearize the equation, we get:

$$\frac{d^2(u-u_0)}{d\phi^2} \approx (-1+6Mu_0)(u-u_0) = -\sqrt{1-12M^2/l^2}(u-u_0) \approx -(1-6M^2/l^2)(u-u_0) \tag{262}$$

where we used the approximation that M^2/l^2 is very small. For Mercury, it is around 3×10^{-8}. (See (266) for its explicit calculation.) Therefore we get:

$$u - u_0 \approx A\cos(\sqrt{1-6M^2/l^2}\phi + \phi_0) \approx A\cos((1-3M^2/l^2)\phi + \phi_0) \tag{263}$$

Here we see that Mercury comes to the closest point to the Sun every time it orbits by the angle $2\pi/(1-3M^2/l^2) \approx 2\pi(1+3M^2/l^2) = 2\pi + 6\pi M^2/l^2$. As this is not equal to 2π, the semi-major axis of the elliptical orbit of Mercury moves by $6\pi M^2/l^2$ radians for every rotation. This is called the "precession of the perihelion of Mercury." The perihelion is the closest point to the Sun in a planetary orbit.

Using the following formula for the eccentricity

$$e = \sqrt{1 - \frac{l^2}{GMa}} = \sqrt{1 - \frac{l^2}{Ma}} \tag{264}$$

where a is the semimajor axis and the following formula that gives the period in terms of the semimajor axis

$$T = 2\pi \frac{a^{3/2}}{GM} = 2\pi \frac{a^{3/2}}{M} \tag{265}$$

we get that the axis of the orbit of Mercury moves by the following angle every rotation

$$6\pi \frac{M^2}{l^2} = 6\pi \frac{Ma}{l^2} \frac{M}{a} = \frac{6\pi}{1-e^2} \frac{4\pi^2 a^2}{T^2} = \frac{24\pi^3 a^2}{T^2(1-e^2)} = \frac{24\pi^3 a^2}{c^2 T^2 (1-e^2)} \tag{266}$$

where in the last step we divided by c^2 to make the whole expression dimensionless, since radians are dimensionless.

According to Wikipedia, the precession of the perihelion for Mercury is given by 575 seconds of arc per century (60 minutes of arc is a degree; 60 seconds of arc is a minute of arc). 532 seconds of the precession are due to the gravitational force of other planets and 43 seconds of arc are due to the relativistic effect which we just calculated. So, if one doesn't count the relativistic effect, there is a discrepancy of about 40 seconds. This discrepancy was discovered by Urbain Le Verrier in 1859. Albert Einstein resolved this discrepancy.

Now we consider our second example: Einstein's prediction of the bending of light due to the Sun. We will closely follow d'Inverno's textbook. First, remember that the line element of light is always lightlike (i.e. $\Delta s^2 = \Delta \tau^2 = 0$). Therefore we cannot use (252). Instead, we should go back to (244), plug in $ds^2 = 0$ and divide by dw^2 on both sides, where w is a parameter that parametrizes the path of the light. Taking steps similar to what we did above, we obtain:

$$\left(1 - \frac{2M}{r}\right) \dot{t} = k \tag{267}$$

$$r^2 \dot{\phi} = l \tag{268}$$

where this time, the dot denotes the w derivative. Then, as before, we get

$$k^2 - \dot{r}^2 - r^2 \dot{\phi}^2 \left(1 - \frac{2M}{r}\right) = 0 \tag{269}$$

$$k^2 - l^2 \left(\frac{du}{d\phi}\right)^2 - l^2 u^2 \left(1 - \frac{2M}{r}\right) = 0 \tag{270}$$

$$\frac{d^2 u}{d\phi^2} + u = 3Mu^2 \tag{271}$$

Now, we have to solve the above equation. As $3M$ is very small, we can perturbatively Taylor-expand the solution u as follows:

$$u = u_0 + 3Mu_1 + (3M)^2 u_2 + \cdots \tag{272}$$

We will consider up to the first power of $3M$ as $3M$ is very small. Now, from this expansion, it is easy to notice that u_0 is the solution when $3M$ is zero. This implies:

$$\frac{d^2 u_0}{d\phi^2} + u_0 = 0 \tag{273}$$

Then the solution is given as follows:

$$u_0 = \frac{1}{D}\sin(\phi - \phi_0) \tag{274}$$

Without loss of generality, we can choose a polar coordinate system that satisfies $\phi_0 = 0$. Then, we have:

$$u_0 = \frac{1}{D}\sin\phi \tag{275}$$

One can actually check that this equation corresponds to a straight line, as it must since the absence of M implies the absence of gravity.

Problem 22. Check this. (Hint[9])

Problem 23. Show that the minimum distance between light and the Sun is D in this case.

Problem 24. Show that light approaches from the point $\phi = 0$ and moves away to the point $\phi = \pi$. (Hint[10]) Notice that in this case, as it goes in a straight line without bending, the values of ϕ when $u = 0$ differ by an angle of π

Now let's plug (272) into (271). We get:

$$\begin{aligned} 0 + 3M\left(\frac{d^2 u_1}{d\phi^2} + 3Mu_1\right) + \cdots &= 3Mu_0^2 + \cdots \\ \frac{d^2 u_1}{d\phi^2} + u_1 &= \frac{\sin^2\phi}{D^2} \end{aligned} \tag{276}$$

It turns out that the solution to this differential equation is given by

$$u_1 = \frac{1}{3D^2}(1 + C\cos\phi + \cos^2\phi) \tag{277}$$

One can check that this is indeed a solution by plugging this into the differential equation (**Problem 25.**).

Summarizing, we have:

$$u \approx \frac{\sin\phi}{D} + \frac{M(1 + C\cos\phi + \cos^2\phi)}{D^2} \tag{278}$$

[9] (r, ϕ) form a polar coordinate system. Transform the above solution into Cartesian coordinate.

[10] When light is infinitely far away from the Sun, $u = 0$.

Now, let's say light comes from $\phi = -\epsilon_1$ and moves away to $\phi = \pi + \epsilon_2$, where ϵ_1 and ϵ_2 are very small, since the path of light is very close to a straight line. If we plug these values into (278), we get:

$$-\frac{\epsilon_1}{D} + \frac{M}{D^2}(2 + C) = 0, \qquad -\frac{\epsilon_2}{D} + \frac{M}{D^2}(2 - C) = 0 \tag{279}$$

As δ, the angle by which the light bends due to the gravitation of the Sun is given by the sum of ϵ_1 and ϵ_2, we can add up the above two equations and obtain:

$$\delta = \epsilon_1 + \epsilon_2 = 4M/D = \frac{4GM}{c^2 D} \tag{280}$$

For the Sun, δ has a certain maximum value, since D cannot be smaller than the radius of the Sun, as the path of light is obstructed by the Sun otherwise. When D is just slightly bigger than the radius of the Sun, δ is about 1.75 seconds of arc. This can be confirmed by measuring the apparent position of stars around the Sun during a total eclipse. When an eclipse takes place, as it is dark, we can see some stars including the ones near the Sun on the sky. As the lights coming from these stars bend due to gravitation of the Sun, the apparent position of the stars during such an eclipse will differ from their actual position. The former can be found by photographing the stars during a total eclipse and the latter at night six months before the eclipse. Then one can compare the pictures with one another. These observations were done by a British team of scientists during a total eclipse in 1919, and, when their analysis of the photographs several months later in November showed that the light had bent by the angle Einstein had predicted, he became a super star.

25 Gravitational waves

In 2016, it was announced that LIGO (Laser Interferometer Gravitational-Wave Observatory) had detected gravitational waves the previous year. The public interest in gravitational waves surged. In this section, we will introduce gravitational waves.

We have already presented an example of solutions for Einstein equation, namely, Schwarzschild solution. Gravitational waves are also solutions for Einstein equation. However, unlike Schwarzschild solution, which is an *exact* solution for Einstein's equation, we will now consider a type of gravitational waves that are only *approximate* solutions for Einstein's equation. The strategy is as follows. We know that a flat space (i.e. the one with flat metric) is a solution to Einstein's equation in empty space (i.e. no matter). This is so because Einstein tensor for the flat metric, as well as the energy-momentum tensor for empty space is zero. Now, we will consider a case in which the metric slightly deviates from the flat-Minkowskian metric, but still a solution to Einstein's equation in empty space. In other words, we will solve Einstein's equation in such

a case, assuming that the deviation from the flat-Minkowskian metric is very small; we will only keep track of the terms which are of the first order in the deviation from the flat-Minkowskian metric, neglecting higher order terms. In other words, we will solve "linearized" Einstein's equation. Then, we will show that the deviation from the flat metric travels at the speed of light, and obtain its polarization states. In other words, we will find which components of the metric deviation can't be "gauged away" to become zero (removed by a suitable gauge choice); we will find what the actual physical metric deviations are.

As advertised, let's write the metric as

$$g_{ab} = \eta_{ab} + h_{ab} \tag{281}$$

where η_{ab} is the flat-Minkowskian metric given by

$$\eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{282}$$

and h_{ab} is the deviation from the flat metric called "metric perturbation." As mentioned, we assume that the metric perturbation is very small, i.e., $|h_{ab}| \ll 1$. Of course, h_{ab} is symmetric as the metric is symmetric.

From now on, indices will be raised and lowered by η_{ab} and η^{ab} instead of g_{ab} and g^{ab}, even though it wouldn't make much difference in linearized gravity as η_{ab} is very close to g_{ab}. Thus, we have

$$h^{bc} = \eta^{bd}\eta^{ce}h_{de} \tag{283}$$

Given this, you can check, the inverse metric is given by

$$g^{bc} = \eta^{bc} - h^{bc} + \mathcal{O}(h^2) \tag{284}$$

Problem 34. Check this by multiplying (281) and (284) to obtain

$$g_{ab}g^{bc} = \delta^c_a + \mathcal{O}(h^2) \tag{285}$$

If we denote the spacetime derivatives by commas, for the Christoffel symbol, we get

$$\begin{aligned} \Gamma^a_{bc} &= \frac{1}{2}g^{ad}(g_{dc,b} + g_{db,c} - g_{bc,d}) \\ &= \frac{1}{2}\eta^{ad}(h_{dc,b} + h_{db,c} - h_{bc,d}) \\ &= \frac{1}{2}(h^a{}_{c,b} + h^a{}_{b,c} - h_{bc,}{}^a) \end{aligned} \tag{286}$$

From this Christoffel symbol, one can get the Riemann tensor. Remember that the Riemann tensor can be written as

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \mathcal{O}(\Gamma^2) \tag{287}$$

In our case, we ignore the term $\mathcal{O}(\Gamma^2)$, because it corresponds to a higher order term $\mathcal{O}(h^2)$. In other words, we only consider terms linear in h. This yields

$$R_{abcd} = \frac{1}{2}(h_{ad,bc} + h_{bc,ad} - h_{ac,bd} - h_{bd,ac}) \tag{288}$$

Problem 35. Check that the above Riemann tensor satisfies the following properties

$$R_{abcd} = R_{cdab} = -R_{abdc} = -R_{bacd} \tag{289}$$

Actually, one can pretty much determine the form of the Riemann tensor from these properties. Since it has four indices, while h has two indices, the extra two indices should be in the form of derivatives. So, one can guess say, a term such as $h_{ab,cd}$ must be present. However, this wouldn't make sense as $h_{ab,cd}$ is symmetric with respect to c and d, while the Riemann tensor is antisymmmetric with respect to c and d. Therefore, we guess again with another term, say $h_{ad,bc}$. Since it is antisymmmetric with respect to c and d we should also have $-h_{ac,bd}$. Therefore, so far, we have two terms $h_{ad,bc} - h_{ac,bd}$. To make this antisymmetric with respect to a and b, we add two terms $-h_{bd,ac} + h_{bc,ad}$. Summarizing, we get

$$R_{abcd} \propto (h_{ad,bc} - h_{ac,bd} - h_{bd,ac} + h_{bc,ad}) \tag{290}$$

Of course, we can't guess the overall proportionality constant but it is still good.

Now, for future convenience, let's define

$$h \equiv \eta^{ab} h_{ab} = h^b{}_b \tag{291}$$

and the d'Alembertian operator

$$\Box = \eta^{ab} \partial_a \partial_b = \partial^a \partial_a \tag{292}$$

Problem 36. Check that $\Box\phi = 0$ is a partial differential equation for wave ϕ traveling at the speed of light.

Problem 37. Check that the Ricci tensor and the Ricci scalar are given by

$$R_{bd} = \frac{1}{2}(\partial_b \partial^c h_{cd} + \partial_d \partial^c h_{cb} - \partial_b \partial_d h - \Box h_{ba}) \tag{293}$$

$$R = \partial^b \partial^d h_{bd} - \Box h \tag{294}$$

Thus, the Einstein tensor is given by

$$G_{bd} = \frac{1}{2}(\partial_b \partial^c h_{cd} + \partial_d \partial^c h_{cb} - \partial_b \partial_d h - \Box h_{ba} - \eta_{bd}(\partial^a \partial^c h_{ac} - \Box h)) \tag{295}$$

Now, let's define "trace-reversed" metric perturbation $\bar{h}_{ab}$ by

$$\bar{h}_{ab} \equiv h_{ab} - \frac{1}{2}\eta_{ab}h \tag{296}$$

It is called trace-reversed, because $\bar{h} \equiv \bar{h}^b{}_b$, the trace of $\bar{h}_{ab}$ is given by $-h$. (**Problem 38.** Check this!)

Of course, if we "trace-reverse" trace-reversed metric perturbation once more, we get the original metric perturbation (**Problem 39.**)

$$h_{ab} = \bar{h}_{ab} - \frac{1}{2}\eta_{ab}\bar{h} \tag{297}$$

Then,

$$G_{bd} = \frac{1}{2}\left[\partial_b(\partial^c\bar{h}_{cd}) + \partial_d(\partial^c\bar{h}_{cb}) - \Box\bar{h}_{bd} - \eta_{bd}\partial^a(\partial^c\bar{h}_{ac})\right] \tag{298}$$

In empty space, $G_{bd} = 0$. Thus, we obtain a solution to the Einstein equation in empty space $\Box\bar{h}_{bd} = 0$ if

$$\partial^c\bar{h}_{cb} = 0 \tag{299}$$

is satisfied. This gauge condition, variously called the Einstein, Lorentz, de Donder, Hilbert or Fock gauge, can be satisfied by a gauge fixing (i.e. choosing a suitable gauge). So, let's talk about gauge fixing. We have already considered the change of coordinate in (191), which we reproduce here with a slightly notation.

$$x'^a = x^a + \xi^a \tag{300}$$

Under this coordinate transformation, (193) in our situation becomes

$$h'_{ba} = h_{ba} - \partial_a\xi_b - \partial_b\xi_a \tag{301}$$

where we have neglected the higher order term $\mathcal{O}(\xi h)$.

Problem 40. Check that under this gauge transformation the Riemann tensor, i.e., (288) remains the same.

Under this gauge transformation, we have

$$\bar{h}'_{ba} = \bar{h}_{ba} - \partial_a\xi_b - \partial_b\xi_a + \eta_{ba}\partial^c\xi_c \tag{302}$$

$$\partial^b\bar{h}'_{ba} = \partial^b\bar{h}_{ba} - \Box\xi_a \tag{303}$$

Therefore, in order to satisfy the gauge condition

$$\partial^b\bar{h}'_{ba} = 0 \tag{304}$$

we have to choose ξ_a that satisfies

$$\Box\xi_a = \partial^b\bar{h}_{ba} \tag{305}$$

We can always find such a solution. Remember how we found the solution to Poisson's equation in our earlier article "Poisson's equation." The only difference is that we now have the 4-d version of Laplacian, namely, d'Alembertian.

Summarizing, we have (304) and

$$\Box \bar{h}'_{bd} = 0 \tag{306}$$

Now, to facilitate our analysis, let's express the trace-reversed metric perturbation in Fourier components. $\Box \bar{h}'_{ab} = 0$ implies

$$\bar{h}'_{ab} = \int d^4k \, H'_{ab} e^{ik_c x^c} \tag{307}$$

where $k^c k_c = 0$. To learn more about the form of $\bar{h}'_{ab}$, let's be more specific. Let's say that plane gravitational waves moves in the positive z-direction. Then, we can write

$$\bar{h}'_{ab} = \bar{h}'_{ab}(z - ct) \tag{308}$$

In other words, $\bar{h}'_{ab}$ only depends on $(z - ct)$. This implies

$$k_0 = k, \quad k_1 = 0, \quad k_2 = 0, \quad k_3 = -k \tag{309}$$

as we have

$$\bar{h}'_{ab}(z - ct) = \int dk \, H'_{ab} e^{-ik(z-ct)} \tag{310}$$

From now on, we will only consider one mode, and write the above relation as

$$\bar{h}'_{ab} = H'_{ab} e^{ik_c x^c} = H'_{ab} e^{-ik(z-ct)} \tag{311}$$

as it is cumbersome to write $\int dk$ every time. Then, the conidtion $\partial^b \bar{h}'_{ab} = 0$ becomes

$$k^b H'_{ab} = 0 \tag{312}$$

At this point, we must ask how many independent H'_{ab} components there are. H'_{ab} is symmetric, so there are 10 components, but (312) imposes 4 conditions ($a = 0, 1, 2, 3$). Therefore, there are 6 components left. As these conditions impose $H'_{a0} = H'_{a3}$, we can write

$$H'_{ab} = \begin{pmatrix} H'_{00} & H'_{01} & H'_{02} & H'_{00} \\ H'_{01} & H'_{11} & H'_{12} & H'_{01} \\ H'_{02} & H'_{12} & H'_{12} & H'_{02} \\ H'_{00} & H'_{01} & H'_{02} & H'_{00} \end{pmatrix} \tag{313}$$

However, it turns out that we can further reduce the number of components by using gauge transformation once more.

Let's do this. ξ_a that satisfies (305) is not unique as we can add to ξ_a a function χ_a that satisfies $\Box \chi_a = 0$. In other words, a new $\xi_a^{\text{new}} = \xi_a + \chi_a$ also satisfies (305),

namely, $\Box\xi_a^{\text{new}} = \partial^b \bar{h}_{ba}$. In other words, for χ_a that satisfies $\Box\chi_a = 0$, if we transform the coordinate as

$$x''^a = x'^a + \chi^a \tag{314}$$

we still have

$$\partial^b \bar{h}''_{ba} = 0 \tag{315}$$

and thus,

$$\Box\bar{h}''_{ab} = 0 \tag{316}$$

Just like (302), we have

$$\bar{h}''_{ba} = \bar{h}'_{ba} - \partial_a\chi_b - \partial_b\chi_a + \eta_{ba}\partial^c\chi_c \tag{317}$$

As we want the final metric h''_{ba} to only depend on $(z-ct)$ as well, $\bar{h}''_{ba}$ must only depend on $(z-ct)$. Then, it is easy to see that χ_b must only depend on $(z-ct)$ as well. Thus, we can write

$$\chi_b = -iC_b e^{ik_c x^c} \tag{318}$$

where k_c is again given by (309). You can also check that our earlier condition $\Box\chi_b = 0$ is automatically satisfied.

Given this, if we use the following notation

$$\bar{h}''_{ab} = H''_{ab} e^{ik_c x^c} \tag{319}$$

We have

$$H''_{ab} = H'_{ab} - k_a C_b - k_b C_a + \eta_{ab} k^c C_c \tag{320}$$

Of course, you can once again check that, if $k^b H'_{ab}$ is assumed, this implies (**Problem 41.**)

$$k^b H''_{ab} = 0 \tag{321}$$

which is (315).

(320) can be represented as

$$H''_{ab} = \begin{pmatrix} H'_{00} & H'_{01} & H'_{02} & H'_{00} \\ H'_{01} & H'_{11} & H'_{12} & H'_{01} \\ H'_{02} & H'_{12} & H'_{12} & H'_{02} \\ H'_{00} & H'_{01} & H'_{02} & H'_{00} \end{pmatrix} - k \begin{pmatrix} C_0 - C_3 & C_1 & C_2 & C_0 - C_3 \\ C_1 & C_0 + C_3 & 0 & C_1 \\ C_2 & 0 & C_0 + C_3 & C_2 \\ C_0 - C_3 & C_1 & C_2 & C_0 - C_3 \end{pmatrix} \tag{322}$$

Thus, we see that we can remove H'_{01} and H'_{02} by kC_1 and kC_2, and H'_{00} by $k(C_0 - C_3)$. We can also remove the sum of H'_{11} and H'_{22} by $2k(C_0 + C_3)$. In other words, out of six independent components, we can remove four of them by properly choosing C_0, C_1, C_2, C_3. Then, we are left with

$$H''_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H''_{11} & H'_{12} & 0 \\ 0 & H'_{12} & -H''_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{323}$$

Summarizing, $\bar{h}''_{ab}$ has only two independent components: $\bar{h}''_{12}$ and $\bar{h}''_{11}$. Now, we can "trace-reverse" $\bar{h}''_{ab}$ to obtain our final metric h''_{ab}. We obtain

$$h''_{ab} = \bar{h}''_{ab} \tag{324}$$

because the trace is 0.

In conclusion, the only non-zero components of h are $h_{11}, h_{12}, h_{21}, h_{22}$, and there are only two independent components, as we have $h_{12} = h_{21}$, and $h_{22} = -h_{11}$. In other words, gravitational waves have two polarizations.

As a side note, let's think about why we could not eliminate these quantities using the further gauge conditions. It's because (320) implies

$$H''_{12} = H'_{12}, \qquad H''_{11} - H''_{22} = H'_{11} - H'_{22} \tag{325}$$

In other words, $\bar{h}_{12}$ and $\bar{h}_{11} - \bar{h}_{22}$ don't change under the gauge transformation.

Problem 42. Graviton is the particle that carries gravitational waves as much as photon is the particle that carries electromagnetic waves. In other words, gravitons mediate gravity while photons mediate electromagnetic force. We just have seen that there are two degrees of freedom in graviton. Show that graviton has spin 2 by showing that these two components we found rotate by 2θ under the rotation of angle θ around z-axis.

Now, let's visualize the effect of these two polarizations. First, the $h_{22} = -h_{11}$ component. When $h_{12} = 0$, the line element is given by

$$d\tau^2 = dt^2 - (1 - h_{11})dx^2 - (1 + h_{11})dy^2 - dz^2 \tag{326}$$

As h_{11} is oscillating, the distance (i.e. the proper distance) between two fixed points (i.e. fixed $\Delta x, \Delta y, \Delta z$) oscillates when gravitational waves are passing by. In particular, the distance Δs is given by

$$\Delta s^2 = (1 - h_{11})\Delta x^2 + (1 + h_{11})\Delta y^2 + \Delta z^2 \tag{327}$$

First, we see that the distance along z-axis doesn't change, as $\Delta x = \Delta y = 0, \Delta z = L$ means $\Delta s = L$, a fixed number. This shows the transverse nature of gravitational waves, if you remember that the gravitational waves in our example are moving in z-direction. Second, we see that when the distance aligned along x-axis is contracted, the distance

aligned along y-axis is expanded and vice versa. For example, if $\Delta x = L, \Delta y = \Delta z = 0$, we have $\Delta s = \sqrt{1-h_{11}}L = (1-h_{11}/2)L$. while $\Delta x = \Delta z = 0, \Delta y = L$ yields $\Delta s = \sqrt{1+h_{11}}L = (1+h_{11}/2)L$. When $(1-h_{11}/2)L$ is big, $(1+h_{11}/2)L$ is small and vice versa. See the figure available at `https://commons.wikimedia.org/wiki/File:GravitationalWave_PlusPolarization.gif`. It is known as + polarization, the distance oscillates along x and y directions. Notice also that the distance aligned along the line 45° from x-axis and y-axis do not change. They just stay there, this is so, as $\Delta x = \Delta y = L, \Delta z = 0$ yields $\Delta s = \sqrt{2}L$ and $\Delta x = L, \Delta y = -L, \Delta z = 0$ yields $\Delta s = \sqrt{2}L$.

Now, the second polarization, $h_{12} = h_{21}$ component. When $h_{11} = -h_{22} = 0$, the line element is given by

$$d\tau^2 = dt^2 - dx^2 + 2h_{12}dx\,dy - dy^2 - dz^2 \tag{328}$$

As h_{12} is oscillating, the distance between two fixed points oscillates when gravitational waves are passing by. In particular, the distance Δs is given by

$$\Delta s^2 = \Delta x^2 - 2h_{12}dx\,dy + \Delta y^2 + \Delta z^2 \tag{329}$$

First, as before, we see that the distance along z-axis doesn't change. This again shows the transverse nature of gravitational waves. Second, the distances aligned along x-axis or y-axis don't change as $\Delta x = L, \Delta y = \Delta z = 0$ yields $\Delta s = L$ and $\Delta x = \Delta z = 0, \Delta y = L$ yields $\Delta s = L$. However, the distances aligned along the line 45° from x-axis and y-axis do change. For example, $\Delta x = \Delta y = L, \Delta z = 0$ yields $\Delta s = (1-h_{12}/2)\sqrt{2}L$ while $\Delta x = L, \Delta y = -L, \Delta z = 0$ yields $\Delta s = (1+h_{12}/2)\sqrt{2}L$. Here, we see that if the former is contracted the latter is expanded and vice versa. See the figure available at `https://commons.wikimedia.org/wiki/File:GravitationalWave_CrossPolarization.gif`. This is known as × polarization. The distances oscillate along the lines 45° rotated from x and y direction.

Well, one can actually see that if you rotate × polarization by 45°, you get + polarization. Let's see this. 45° rotation means

$$x' = \cos 45^\circ\, x + \sin 45^\circ y, \qquad y' = -\sin 45^\circ\, x + \cos 45^\circ y \tag{330}$$

then, we have

$$\Delta s^2 = (1-h_{12})\Delta x'^2 + (1+h_{12})\Delta y'^2 + \Delta z^2 \tag{331}$$

This is exactly (327), except for the fact that we now have x', y', and h_{12} instead of x, y, and h_{11}.

In our earlier article "History of Astronomy from the early 20th century to the early 21st century," we talked about the LIGO detector that detected gravitational waves in

2015. We will not repeat the information given in that article, and just mention that the gravitational waves from two merging black holes which LIGO detected in 2015 had the frequency of about 150 Hz, as you can easily estimate from the figure in that article.

Problem 43. In Section 22, we related g_{00}, the time time component of metric with the Newtonian potential in the Newtonian limit. In this problem, we will obtain the space space component of the metric in the Newtonian limit. Remember that in the gauge $\partial^c \bar{h}_{cb} = 0$, we have

$$G_{bd} = -\frac{1}{2}\Box \bar{h}_{bd} \tag{332}$$

Thus, the Einstein equation implies

$$-\frac{1}{2}\Box \bar{h}_{00} = 8\pi G\rho \tag{333}$$

By comparing this with the Poisson equation $\nabla^2\Phi = 4\pi G\rho$, we obtain $\bar{h}_{00} = 4\Phi$. In the Newtonian limit, we have

$$T_{00} \gg T_{0i} \gg T_{ij}, \text{ which implies } G_{00} \gg G_{0i} \gg G_{ij} \tag{334}$$

where i and j denote the spatial component. Thus, we have

$$\bar{h}_{0i} \approx \bar{h}_{ij} \approx 0 \tag{335}$$

which implies $\bar{h} = 4\Phi$. By using (297), we thus obtain $h_{00} = 2\Phi$. By similarly obtaining h_{ij}, show that the metric can be written in the Newtonian limit as follows:

$$d\tau^2 = (1 + 2\Phi)dt^2 - (1 - 2\Phi)(dx^2 + dy^2 + dz^2) \tag{336}$$

26 Vector field and Lie bracket

So far, we expressed a vector field by an object with one upper index, for example, X^a. However, there is an equivalent way of expressing a vector field, more commonly used in math textbooks for which the vector is simply expressed as X. There is no index, and this type of notation is called "index-free notation." Then what is the relation between X and X^a? It is given as follows:

$$X = X^a \partial_a \tag{337}$$

Here, we see that ∂_a serves as the basis. Written this way, a vector field X can be applied to a scalar function f as follows:

$$Xf = X^a \partial_a f \tag{338}$$

In the language of a freshman calculus course, the above is equal to $\vec{X} \cdot \nabla f$. So Xf represents the change of the function f along the vector field $\vec{X}$.

Of course, it goes without saying that this value must be independent of the coordinates we choose, as Xf is a scalar. This means that X itself should not depend on the coordinate system. This is natural, if we consider that a vector field has an independent meaning free from the coordinate systems, even though it can be expressed differently using different coordinate systems. Actually, we can explicitly check this. In the primed coordinate system, X is expressed as

$$\begin{aligned} X &= X'^i \partial'_i = X'^i \frac{\partial}{\partial x'^i} && (339) \\ &= \frac{\partial x'^i}{\partial x^b} X^b \frac{\partial x^a}{\partial x'^i} \frac{\partial}{\partial x^a} && (340) \\ &= X^b \frac{\partial x'^i}{\partial x^b} \frac{\partial x^a}{\partial x'^i} \frac{\partial}{\partial x^a} && (341) \\ &= X^b \delta^a_b \partial_a = X^a \partial_a && (342) \end{aligned}$$

In conclusion, $X = X'^i \partial'_i = X^a \partial_a$.

Having introduced the index-free notation of a vector, let us introduce the Lie bracket. Actually, you already know Lie bracket. It is the same thing as the commutator defined as follow:

$$[X, Y] = XY - YX \tag{343}$$

The new thing you didn't know is that X and Y can be vector fields. Now let's explicitly find the Lie bracket of two vector fields $X = X^a \partial_a$, $Y = Y^a \partial_a f$. We have

$$\begin{aligned} [X, Y]f &= (XY - YX)f = X(Yf) - Y(Xf) && (344) \\ &= X(Y^a \partial_a f) - Y(X^a \partial_a f) && (345) \\ &= X^b \partial_b (Y^a \partial_a f) - Y^b \partial_b (X^a \partial_a f) && (346) \end{aligned}$$

Problem 44. By using the Leibniz rule and the commutativity of partial derivatives (i.e. $\partial_a \partial_b f = \partial_b \partial_a f$), show that the above equation is equal to

$$[X, Y]f = (X^b \partial_b Y^a - Y^b \partial_b X^a) \partial_a f \tag{347}$$

In conclusion, we have

$$[X, Y]^a = X^b \partial_b Y^a - Y^b \partial_b X^a \tag{348}$$

A natural question you may ask is: Can we replace the partial derivatives in the above equation with covariant derivatives? After all, the covariant derivative is just another type of derivative. While covariant derivative has information about connection, partial derivative doesn't. The very fact that the above expression doesn't have covariant derivatives in it implies that Lie bracket does not "care" about connection. Therefore, we can just as well replace the partial derivative by covariant derivative, just another type of derivative. However, the connection must not have torsion.

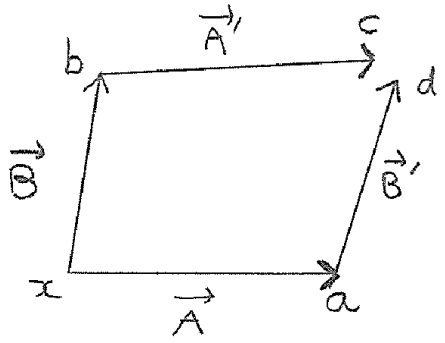

Figure 6: Parallelogram

Problem 45. Show that

$$[X, Y]^a = X^b \nabla_b Y^a - Y^b \nabla_b X^a \tag{349}$$

if the torsion vanishes (i.e. $\Gamma^c_{ab} = \Gamma^c_{ba}$).

Having mentioned torsion, let's find its geometrical meaning. Simply speaking, the torsion vanishes if all the infinitesimal parallelograms "close." Let me explain what I mean here. See Fig. 6. in which an infinitesimal vector $\vec{A}$ is parallel-transported along another infinitesimal vector $\vec{B}$ (i.e. the path from x to b). After the parallel-transportation, it becomes $\vec{A}'$. The tip of the parallel-transported vector $\vec{A}'$ is at the point c. Similarly, $\vec{B}$ is parallel-transported along the vector $\vec{A}$ (i.e. the path from x to a). After the parallel-transportation, it becomes $\vec{B}'$. The tip of the parallel-transported vector $\vec{B}'$ is at the point d. If the position of c coincides with the position of d, the torsion vanishes, as we will see below. That's what I mean by the statement that the infinitesimal parallelograms "close."

Let's check it. We have

$$A'^\mu = A^\mu - \Gamma^\mu_{\alpha\beta} A^\alpha B^\beta \tag{350}$$

$$B'^\mu = B^\mu - \Gamma^\mu_{\beta\alpha} B^\beta A^\alpha \tag{351}$$

So, the vector pointing from c to d is given by

$$(A^\mu + B'^\mu) - (B^\mu + A'^\mu) = (\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha}) A^\alpha B^\beta \tag{352}$$

The torsion tensor is defined by

$$T^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha} \tag{353}$$

It goes without saying that the torsion tensor is antisymmetric with respect to the lower two indices. So the vector pointing from c to d is given by

$$T^\mu_{\alpha\beta} A^\alpha B^\beta \tag{354}$$

It vanishes when the torsion is zero.

I won't say more about torsion, not only because I do not know much more about torsion, but also because you can look at other sources if you want to learn more about it. Look for "Einstein-Cartan theory."

27 Lie derivative

In an earlier article, we introduced the covariant derivative. In this article, we will introduce yet another type of derivative called the "Lie derivative."

Consider the following change of coordinate, for an infinitesimal ξ^a

$$x^a = x'^a + \xi^a(x') \tag{355}$$

Then, a tensor $T_{ab}(x)$ transforms as

$$T_{ab}(x) = \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b}T'_{cd}(x') \tag{356}$$

As ξ^a is infinitesimal, (355) can be re-expressed as

$$x'^a = x^a - \xi^a(x) \tag{357}$$

Problem 46. Obtain the following expression. (Hint[11])

$$T'_{ab}(x) - T_{ab}(x) = \xi^c\partial_c T_{ab} + \partial_a\xi^c T_{cb} + \partial_b\xi^d T_{ad} \tag{358}$$

Now, we can define the Lie derivative. It is defined by the right-hand side of the above expression. i.e.,

$$T'_{ab}(x) - T_{ab}(x) = \mathcal{L}_\xi T_{ab} \tag{359}$$

where we have (355) for an infinitesimal ξ^a.

Notice that $\mathcal{L}_\xi T_{ab}$ is still a rank 2 tensor. By extension, the Lie derivative of a rank n tensor is still a rank n tensor. In other words, the tensor type is preserved.

Problem 47. Check

$$\mathcal{L}_\xi f = \xi^c\partial_c f = \xi f \tag{360}$$

where f is a scalar. This shows that the Lie derivative of a scalar coincides with the usual derivative.

Problem 48. Check

$$\mathcal{L}_\xi T^{ab} = \xi^c\partial_c T^{ab} - T^{cb}\partial_c\xi^a - T^{ac}\partial_c\xi^b \tag{361}$$

Problem 49. Check

$$\mathcal{L}_X Y = [X, Y] \tag{362}$$

[11] See how (192) was derived.

where X and Y are vectors. In other words,

$$\mathcal{L}_X Y^a = X^b \partial_b Y^a - Y^b \partial_b X^a. \tag{363}$$

Problem 50. Check

$$\mathcal{L}_X Y_a = X^b \partial_b Y_a + Y_b \partial_a X^b. \tag{364}$$

Similarly, it is also easy to check for a general tensor that the Lie derivative is given by

$$\mathcal{L}_X T^{a\cdots}_{b\cdots} = X^c \partial_c T^{a\cdots}_{b\cdots} - T^{c\cdots}_{b\cdots} \partial_c X^a - \cdots + T^{a\cdots}_{c\cdots} \partial_b X^c \tag{365}$$

Note that the partial derivatives in the above equations can be replaced by the covariant derivatives if the torsion vanishes, from the same reason as we could so for Lie bracket.

So far, we have introduced the Lie derivative heuristically. Actually, the Lie derivative can be defined more rigorously. The claim is that the derivative that satisfies the following four axioms must necessarily be the Lie derivative. We will not prove this claim.

- **Axiom 1.** The Lie derivative of a scalar field is given by

$$\mathcal{L}_X f = X f = X^a \partial_a f \tag{366}$$

- **Axiom 2.** The Lie derivative obeys the Leibniz rule for a tensor product. For example,

$$\mathcal{L}_X (A^a B_{bc}) = (\mathcal{L}_X A^a) B_{bc} + A^a (\mathcal{L}_X B_{bc}). \tag{367}$$

- **Axiom 3.** The Lie derivative obeys the Leibniz rule for contraction. For example,

$$\mathcal{L}_X (A^a B_{ab}) = (\mathcal{L}_X A^a) B_{ab} + A^a (\mathcal{L}_X B_{ab}). \tag{368}$$

- **Axiom 4.** The Lie derivative commutes with exterior derivative on functions.

$$[\mathcal{L}_X, d] = 0 \tag{369}$$

  In other words,

$$\mathcal{L}_X df = d(\mathcal{L}_X f). \tag{370}$$

28 Killing vector field

A Killing vector field ξ is a vector field that satisfies

$$\mathcal{L}_\xi g_{ab} = 0 \tag{371}$$

The Killing vector is named after the German mathematician, Wilhelm Killing. In components, the above expression is equal to

$$\xi^e \partial_e g_{ab} + g_{ad}\partial_b \xi^d + g_{db}\partial_a \xi^d = \nabla_a \xi_b + \nabla_b \xi_a = 0 \tag{372}$$

We have already encountered similar expressions in Sec. 20.

If the metric coefficients $g_{\mu\nu}$ don't depend on x^i for a certain i, then $K^\mu = \delta^\mu_i$ is automatically a Killing vector. For example, if the metric doesn't depend on time, $K^\mu = (1, 0, 0, 0)$ is a Killing vector, and if it doesn't depend on x^3, $K^\mu = (0, 0, 0, 1)$ is a Killing vector.

Problem 51. Using the first expression of (372), show that $K^\mu = \delta^\mu_i$ is a Killing vector, if the metric doesn't depend on x^i.

For every Killing vector, there is a quantity conserved along geodesics. If the proper time is given by τ, the claim is

$$\frac{d}{d\tau}\left(K_\mu P^\mu\right) = 0 \tag{373}$$

where P^μ is the four-momentum of the particle. In other words, $K_\mu P^\mu$ is constant along geodesics.

Let's prove this. The above equation can be written as

$$\frac{dx^\nu}{d\tau}\partial_\nu\left(K_\mu m\frac{dx^\mu}{d\tau}\right) = m\frac{dx^\nu}{d\tau}\nabla_\nu\left(K_\mu\frac{dx^\mu}{d\tau}\right) \tag{374}$$

$$= m\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau}\nabla_\nu K_\mu + mK_\mu\frac{dx^\nu}{d\tau}\nabla_\nu\left(\frac{dx^\mu}{d\tau}\right) \tag{375}$$

From (115) and (116), the second term in the right-hand side is 0. Then the above expression can be written as

$$m\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau}\nabla_\nu K_\mu = \frac{m}{2}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau}(\nabla_\mu K_\nu + \nabla_\nu K_\mu) = 0 \tag{376}$$

where we used a change of variables on the dummy indices (see Problem 1 of "Einstein summation convention," if you don't understand this). This completes the proof.

Problem 52. Check for a flat Minkowski metric

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 = \eta_{ab}dx^a dx^b \tag{377}$$

∂_t, ∂_x, ∂_y, ∂_z are Killing vectors. Check also that $x_a\partial_b - x_b\partial_a$ for any a, b is also a Killing vector. For example, when $a = 0, b = 1$ we have $x\partial_t + t\partial x$, and when $a = 1, b = 2$ we have $-x\partial_y + y\partial_x$. This should remind you of the Lorentz transformation. The former corresponds to the Lorentz generator of boost along the x direction, and the latter corresponds to the Lorentz generator of rotation around the z axis. So, there are total of ten Killing vectors in flat 4-d space. Of course, this is true for the flat space expressed

in other forms of metric, as the notion of Killing vector should not depend on the metric, if it describes the same spacetime. For example,

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{378}$$

should also have ten Killing vectors. Of course, except for the two obvious Killing vectors ∂_t and ∂_ϕ, it would be a little bit cumbersome to find other Killing vectors by explicitly using $\mathcal{L}_K g_{ab} = 0$. An easier way would be coordinate transforming the Killing vectors we found for (377).

Problem 53. Show that for the metric (377) the conserved quantity for the Killing vector (0,1,0,0) (i.e., ∂_x) corresponds to the x-momentum, and the one for $(0, y, -x, 0)$ (i.e., $-x\partial_y + y\partial_x$) corresponds to the z component of angular momentum.

Problem 54. Check that (1,0,0,0) and (0,0,0,1) are Killing vector fields for the Schwarzschild metric. Then using these Killing vector fields, derive (253) and (254).

29 A glimpse at black hole thermodynamics

In 1974, by considering quantum effects, Hawking showed that a black hole can radiate light with its spectrum being Planck's radiation spectrum. In particular, Hawking showed that the temperature of a black hole corresponds to $\frac{1}{8\pi M}$ where M is the mass of the black hole. (We will show this in the next section.) Given this, we will derive the black hole entropy for case of a Schwarzschild black hole, as it is the simplest case and can be generalized easily. To this end, consider the well-known following thermodynamic identity:

$$dQ = TdS \tag{379}$$

Since Q is the energy, it corresponds to the mass M. Also, from Section 23, we know that $2M = r_s$ where r_s is the radius of the black hole. Therefore, we have:

$$dQ = dM = \frac{dr_s}{2} = \frac{dS}{8\pi M} = \frac{dS}{4\pi r_s} \tag{380}$$

$$S = \pi r_s^2 \tag{381}$$

However, A, the black hole horizon area is given by $4\pi r_s^2$, since the spatial part of the Schwarzschild metric (241) restricted to $r = r_s$ is

$$ds^2 = dr_s^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{382}$$

which is exactly the metric of 2-dimensional space on a 2-sphere with radius r_s. Therefore, we conclude:

$$S = \frac{A}{4} \tag{383}$$

This relation is known as the "Bekenstein-Hawking entropy" and holds for other types of black holes as well, at least to the leading order in A.

30 Derivation of Hawking temperature

In this section, as promised, we will derive $T = 1/8\pi M$ using a trick. You need to know statistical mechanics and quantum field theory well to concretely understand the justification behind this trick, but it is good to know it, as it is the easiest way to calculate the black hole temperature, given its metric. To this end, this section closely follows pages 562∼563 of *String theory and M-theory* by Becker, Becker and Schwarz.

Recall that the partition function in statistical mechanics is given as follows:

$$Z = \mathrm{Tr}(e^{-\beta H}) \tag{384}$$

where β is given by $1/kT$. In the natural units in which we set $k = 1$, this is simply given by $1/T$.

In our earlier article "Feynman path integral," we have seen that partition function for the Euclideanized theory is given in terms of $\tau = it$, the Euclideanized time, which has period β. Therefore, if we find the period for τ from the metric, our job is done. To this end, let's define ρ by the following formula:

$$r = r_s(1 + \rho^2) \tag{385}$$

So, when ρ is small, we can examine the vicinity of the black hole horizon. In this limit (241) becomes

$$ds^2 \approx r_s^2 \left(d\rho^2 + \rho^2 \left(\frac{d\tau}{2r_s} \right)^2 + \frac{1}{4}(d\theta^2 + \sin^2\theta d\phi^2) \right) \tag{386}$$

The first two terms have the form of the metric for a flat plane in polar coordinates, if we identify $\frac{d\tau}{2r_s}$ with $d\Theta$, where Θ is the angular coordinate. As Θ has period 2π, τ has period $4\pi r_s$ which, in turn, is equal to $8\pi M$. Since this is β, the Hawking temperature $T = \beta^{-1}$ is equal to $1/(8\pi M)$.

Summary

- The metric tells you the distance between two points.
- In 3-dimensional flat Cartesian coordinate, the metric is given by

$$ds^2 = dx^2 + dy^2 + dz^2$$

- The same can be expressed in the spherical coordinate as

$$ds^2 = dr^2 + r^2 d\Omega^2$$

  where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$

- The metric tensor g_{ab} is defined by

$$ds^2 = g_{ab}dx^a dx^b$$

- In the flat Minkowski space, the metric tensor is given by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{or} \quad g_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  depending on the convention.

- The Jacobian in 4-dimensional spacetime is given by $\sqrt{-g}$ where g is the determinant of the metric tensor g_{ab}.

- A vector can be expressed as A^a. A dot-product can be expressed as

$$\vec{A} \cdot \vec{B} = g_{ab}A^a B^b$$

- A covector B^a is defined by $B_a = g_{ab}B^b$. This allows us to express a dot-prduct as

$$\vec{A} \cdot \vec{B} = A^a B_a = A_a B^a$$

- The inverse metric g^{ab} is the inverse of the metric tensor g_{ab}. i.e.,

$$g^{ab}g_{bc} = \delta^a_c$$

  This allows us to express a vector in terms of a covector as $A^a = g^{ab}A_b$.

- A (p, q)-rank tensor has p upper indices and q lower indices. It transforms like a tensor product of p vectors and q covectors.

- The Levi-Civita tensor ϵ is defined by the Levi-Civita symbol $\tilde{\epsilon}$ as

$$\epsilon_{abc\cdots} = \sqrt{|g|}\tilde{\epsilon}_{abc\cdots}$$

- The action S can be expressed by a Lagrangian density $\mathcal{L}$ as follows

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$$

- In such a case, we can obtain the Euler-Lagrange equation just as the case with classical mechanics case. We have

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

- The partial derivative of a scalar field (i.e., $\partial_a \phi$) is a covariant covector. However, the partial derivative of a vector (i.e., $\partial_a V^b$) is not a tensor.

- To remedy this situation, we define the covariant derivative as

$$\nabla_c V^a = \partial_c V^a + \Gamma^a_{bc} V^b$$

  Then, the covariant derivative is a tensor.

- A geodesic is a path that minimizes the proper time or the proper distance. We can obtain it by solving the Euler-Lagrange equation for the proper time or the proper distance.

- If you parallel transport a vector V^a along the direction A^c we have

$$A^c \nabla_c V^a = 0$$

- If you parallel transport a vector V^a along its own direction (i.e., V^a) we have

$$V^c \nabla_c V^a = 0$$

  which is exactly the geodesic. By comparing this formula with the geodesic equation one can obtain Γ^a_{bc}

- The covariant derivative of a co-vector is given by

$$\nabla_c B_a = \partial_c B_a - \Gamma^b_{ac} B_b$$

- The Christoffel symbol Γ^a_{bc} can be obtained alternatively from the condition that the covariant derivative of a metric vanishes (i.e., $\nabla_a g_{bc} = 0$) and the torsion-free condition (i.e., $\Gamma^a_{bc} = \Gamma^a_{cb}$).

- The Riemann tensor is defined by (in the absence of torsion)

$$R^a{}_{bcd} V^b = \nabla_c \nabla_d V^a - \nabla_d \nabla_c V^a$$

  As the right-hand side is a tensor, the Riemann tensor is a tensor.

- The Riemann tensor tells you the difference of parallel-transported vector along two different paths.

- The Riemann tensor is antisymmetric with respect to the first two indices and the last two indices. i.e.,

$$R_{abcd} = -R_{bacd}, \qquad R_{abcd} = -R_{abdc}$$

  It is also invariant under the change of the whole first two indices and the whole last two indices. i.e,

$$R_{abcd} = R_{cdab}$$

- The sum of the cyclic permutation of the last three indices of the Riemann tensor is zero. i.e,

$$R_{dabc} + R_{dbca} + R_{dcab} = 0$$

It also satisfies the Bianchi identity, which can be obtained by the cyclic permutation of a, b, c of $\nabla_a \nabla_b \nabla_c V^d$.

- The Ricci tensor is defined by

$$R_{ab} = R^c{}_{acb}$$

It is symmetric. i.e., $R_{ab} = R_{ba}$.

- The Ricci scalar is defined by

$$R = g^{ab} R_{ab}$$

- The Einstein tensor is defined by

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$$

The Bianchi identity implies $\nabla^a G_{ab} = 0$.

- The Einstein-Hilbert action is given by

$$S = \int d^4x \sqrt{-g} R$$

- One can derive the following Einstein equation from the Einstein-Hilbert action.

$$G_{ab} = 8\pi G T_{ab}$$

where T_{ab} is the energy-momentum tensor.

- The energy-momentum conservation can be expressed as

$$\nabla^a T_{ab} = 0$$

This fits nicely with the Einstein equation, because both sides are zero if we take ∇^a on both sides of the Einstein equation.

- For perfect fluid, the energy momentum tensor is given by

$$T^{ab} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

- In the Newtonian limit, we have

$$\frac{dx^0}{d\tau} \gg \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau}$$

and

$$\partial_0 \ll \partial_1, \partial_2, \partial_3, \qquad d\tau = cdt$$

In this limit, Einstein equation reduces to the Newtonian gravity.

- Schwarzschild solution is a solution to the Einstein equation $G_{ab} = 0$ except when $r = 0$. The time slows at the place with lower gravitational potential. The rate of time flowing is given by $\sqrt{|g_{tt}|}$.

- The Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

describes the Schwarzschild black hole with mass M. The g_{tt} and g_{rr} components of the Schwarzschild metric change signs when crossing $r = 2M$ where M is the mass of the black hole. $r_s = 2M$ is called the Schwarzschild radius. The region $r = 2M$ is called "black hole horizon." An object inside a black hole (i.e., $r < 2M$) can never escape outside the black hole (i.e., $r > 2M$).

- If we consider a small perturbation around the flat metric and solve the Einstein equation we can obtain an equation for gravitational waves.

- A gravitational wave is a transverse wave (i.e., perpendicular to the travelling direction) that travels at the speed of light. It has two polarizations, and the trace of the gravitational wave is zero.

- In the Newtonian limit, the metric can be written as

$$d\tau^2 = (1 + 2\Phi)dt^2 - (1 - 2\Phi)(dx^2 + dy^2 + dz^2)$$

where Φ is the Newtonian potential.

- The Bekenstein-Hawking entropy is the entropy of a black hole and is given by

$$S = \frac{A}{4}$$

where A is the area of the black hole horizon.

- The Hawking temperature is given by the inverse of the period of the Euclideanized time.

- A vector field in an index-free notation is denoted by X. Its relation with the index notation is given by $X = X^a \partial_a$.

- This notation is independent of coordinates as

$$X = X^a \partial_a = X'^i \partial'_i$$

- When applied to a function, such a vector field acts as a gradient:

$$Xf = X^a \partial_a f = \vec{X} \cdot \nabla f$$

- The Lie bracket for a vector field is similarly defined:

$$[X, Y]f = [XY - YX]f$$

  which implies

$$[X, Y]^a = X^b \partial_b Y^a - Y^b \partial_b X^a$$

  If the torsion vanishes the partial derivatives in the above expression can be replaced by covariant derivatives.

- The Lie derivative of a tensor $T^{a\cdots}_{b\cdots}$ is given by

$$\mathcal{L}_\xi T^{a\cdots}_{b\cdots} = T'^{a\cdots}_{b\cdots}(x) - T^{a\cdots}_{b\cdots}(x)$$

  where $x^c = x'^c + \xi^c$

- The first term in the Lie derivative starts from $\xi^c \partial_c$ and up to sign, each other term is given by the first derivative of ξ^c (such as $\partial_a \xi^c$) multiplied by the concerned tensor with appropriate index. The sign depends on whether the index is upper or lower index.

- $\mathcal{L}_X f = X^a \partial_a f = Xf, \qquad \mathcal{L}_X Y = [X, Y]$

- The partial derivative in the Lie derivative can be replaced by the covariant derivative, if torsion vanishes.

- Killing vector K^μ satisfies

$$\mathcal{L}_K g_{ab} = \nabla_a K_b + \nabla_b K_a = 0$$

- If the metric coefficients don't depend on x^i for a certain i or i's, then ∂_i is automatically a Killing vector.

- If the proper time is given by τ, and K^μ a Killing vector, the quantity

$$K_\mu \frac{dx^\mu}{d\tau}$$

is preserved along geodesics.

Further Reading

The following four books which I recommend were most helpful when preparing this review paper: *Introducing Einstein's Relativity* by d'Inverno, *A First Course in General Relativity* by Schutz, *Spacetime and Geometry: An Introduction to General Relativity* by Sean Carroll, and *General Relativity* by Robert Wald.

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A Proof of $\delta g = -g g_{ab} \delta g^{ab}$

First note that

$$\det g = \tilde{\epsilon}^{abcd} g_{0a} g_{1b} g_{2c} g_{3d} \tag{387}$$

If we differentiate the above determinant with respect to say, g_{1e}. Then, we get

$$\frac{\delta \det g}{\delta g_{1e}} = \tilde{\epsilon}^{aecd} g_{0a} g_{2c} g_{3d} \tag{388}$$

On the other hand, let's calculate the quantity $\det g g^{e1}$. We get,

$$\det g g^{e1} = \tilde{\epsilon}^{abcd} g_{0a} g^{e1} g_{1b} g_{2c} g_{3d} \tag{389}$$

$$= \tilde{\epsilon}^{abcd} g_{0a} \delta^e_b g_{2c} g_{3d} \tag{390}$$

$$= \tilde{\epsilon}^{aecd} g_{0a} g_{2c} g_{3d} \tag{391}$$

which is exactly equal to (388). Thus, we conclude

$$\frac{\delta \det g}{\delta g_{1e}} = \det g g^{e1} \tag{392}$$

More generally, we have

$$\frac{\delta \det g}{\delta g_{fe}} = \det g g^{ef} \tag{393}$$

As an aside, it goes without saying that for a general matrix A we have

$$\frac{\delta \det A}{\delta A_{fe}} = \det A (A^{-1})^{ef} \tag{394}$$

where $(A^{-1})^{gf}A_{fe} = \delta^g_e$.

Using the notation $g = \det g$ and the symmetricity of metric, (393) means

$$\delta g = g g^{ab} \delta g_{ab} \tag{395}$$

Now, notice

$$0 = \delta(4) = \delta\delta^a_a = \delta(g^{ab}g_{ab}) = \delta g^{ab} g_{ab} + g^{ab}\delta g_{ab} \tag{396}$$

which implies

$$g^{ab}\delta g_{ab} = -g_{ab}\delta g^{ab} \tag{397}$$

Thus, (395) becomes

$$\delta g = -g g_{ab}\delta g^{ab} \tag{398}$$

which implies (83).

B A crucial property of tensorial equations

Let's say that in a certain coordinate system the following tensor equation holds:

$$X_{ab} = Y_{ab} \tag{399}$$

Now, let's multiply both sides by $\partial x^a/\partial x'^i$ and $\partial x^b/\partial x'^j$. Then, we have

$$\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j} X_{ab} = \frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j} Y_{ab} \tag{400}$$

Thus, we get

$$X'_{ij} = Y'_{ij} \tag{401}$$

Therefore, we see that (399) implies (401). In other words, if a tensor equation holds in one coordinate system, it holds in all other coordinate systems. We will use this result in Appendix E.

C Transformation law for Γ^a_{cb}

The connection Γ^a_{cb} is not a tensor, as a connection is expressed in terms of the difference between $\nabla_b A^a$, which is a tensor, and $\partial_b A^a$ which is not a tensor. Therefore, Γ^a_{cb} does not transform covariantly. Actually, you can check this explicitly:

Problem 54. Show that the connection transforms as follows (Hint[12]):

$$\Gamma'^j_{kl} = \frac{\partial x^b}{\partial x'^i}\frac{\partial x'^j}{\partial x^a}\frac{\partial x^c}{\partial x'^k}\Gamma^a_{cb} - \frac{\partial x^b}{\partial x'^i}\frac{\partial x^a}{\partial x'^k}\frac{\partial x'^j}{\partial x^b \partial x^a} \tag{402}$$

[12] Use (95).

D Geodesic coordinate

Let's choose an arbitrary point P. Then, we will show that we can always find a coordinate system that the connection at that point vanishes. In other words, we can find a coordinate system that satisfies $\Gamma^a_{bc}(P) = 0$. Such a coordinate system is called "geodesic coordinate." Let's see how we can construct such a coordinate system.

Without loss of generality, let's first choose P to be at the origin. Then, we have $x^a(P) = 0$. Now, consider the following coordinate transformation:

$$x'^a = x^a + \frac{1}{2} Q^a_{bc} x^b x^c \tag{403}$$

Then, we have

$$\frac{\partial x'^a}{\partial x^d} = \delta^a_d + Q^a_{bd} x^d, \qquad \frac{\partial^2 x'^a}{\partial x^d \partial x^e} = Q^a_{de} \tag{404}$$

As $x^d(P) = 0$, the first equation implies

$$\frac{\partial x'^a}{\partial x^d}(P) = \delta^a_d \tag{405}$$

As the inverse matrix of the right-hand side is also the identity matrix, we have

$$\frac{\partial x^d}{\partial x'^a}(P) = \delta^d_a \tag{406}$$

Plugging this into (402), we get

$$\Gamma'^j_{kl}(P) = \Gamma^j_{kl}(P) - Q^j_{kl} \tag{407}$$

Thus, by choosing Q^j_{kl} that satisfies

$$Q^j_{kl} = \Gamma^j_{kl}(P) \tag{408}$$

we can make

$$\Gamma'^j_{kl}(P) = 0 \tag{409}$$

Thus, we have successfully constructed a geodesic coordinate. Notice that a geodesic coordinate doesn't imply that the connection Γ'^j_{kl} vanishes at all points; it is only guaranteed that it vanishes at P. Of course, if the connection vanishes at all points, the manifold is necessarily flat.

We will see the usefulness of the geodesic coordinate in the next section.

E $\sqrt{-g} g^{ab} \delta R_{ab}$ is a total derivative

To this end, we first need to show that $\delta\Gamma^a_{cb}$, the variation of a connection is a tensor. This can be immediately seen from (402), but let's take a different reasoning.

Remember

$$\nabla_b A^a = \partial_b A^a + \Gamma^a_{cb} A^c \tag{410}$$

If we vary Γ^a_{cb} as follows

$$\Gamma^a_{cb} \to \Gamma^a_{cb} + \delta\Gamma^a_{cb} \tag{411}$$

we have

$$\nabla_b A^a \to \nabla_b A^a + \delta\Gamma^a_{cb} A^b \tag{412}$$

In a different coordinate, this corresponds to

$$\nabla'_i A'^j \to \nabla'_i A^j + \delta\Gamma'^j_{ki} A'^k \tag{413}$$

As we have

$$\nabla'_i A'^j = \frac{\partial x^b}{\partial x'^i}\frac{\partial x'^j}{\partial x^a}\nabla_b A^a \tag{414}$$

and

$$\nabla'_i A'^j + \delta\Gamma'^j_{ki} A'^k = \frac{\partial x^b}{\partial x'^i}\frac{\partial x'^j}{\partial x^a}(\nabla_b A^a + \delta\Gamma^a_{cb} A^b) \tag{415}$$

subtracting (414) from (415) we conclude that $\delta\Gamma^a_{cb}$ transforms covariantly. In other words, the difference of two tensors is a tensor.

Now, in geodesic coordinates, we can set $\Gamma^a_{bc} = 0$ locally at a desired point P. Then, we can write

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} \tag{416}$$

at this point. Then, under the variation (411), the above Riemann tensor varies by

$$\delta R^a{}_{bcd} = \partial_c \delta\Gamma^a_{bd} - \partial_d \delta\Gamma^a_{bc} \tag{417}$$

As Γ^a_{bc}s are all zero, at the point P, the right-hand side of the above formula can be re-written as

$$\delta R^a{}_{bcd} = \nabla_c \delta\Gamma^a_{bd} - \nabla_d \delta\Gamma^a_{bc} \tag{418}$$

However, in Appendix B, we proved that a tensor equation holds in all coordinate system, if it holds in one particular coordinate system. Thus, (418) holds in all coordinate systems. (418) is known as "Palatini equaion." Actually Palatini equation can be derived by using straightforward calculations without using a geodesic coordinate.

From (418), we get

$$\delta R_{bd} = \nabla_a \delta\Gamma^a_{bd} - \nabla_d \delta\Gamma^a_{ba} \tag{419}$$

As the covariant derivative of metric is zero, we can write

$$\begin{aligned} \sqrt{-g} g^{bd} \delta R_{bd} &= \sqrt{-g}\nabla_a (g^{bd}\delta\Gamma^a_{bd}) - \sqrt{-g}\nabla_d (g^{bd}\delta\Gamma^a_{ba}) \\ &= \sqrt{-g}\nabla_a (g^{bd}\delta\Gamma^a_{bd} - \sqrt{-g} g^{ba}\delta\Gamma^d_{bd}) \end{aligned} \tag{420}$$

The right-hand side is of the form $\sqrt{-g}\nabla_a v^a$. Now, recall (137). We have

$$\int d^4x\sqrt{-g}\nabla_a v^a = \int d^4x\sqrt{-g}\frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}v^a) = \int d^4x\partial_a(\sqrt{-g}v^a) \tag{421}$$

Indeed, we see that it is a total derivative.

F Proof of $R_{abcd} = R_{cdab}$

$$R_{abcd} + R_{acdb} + R_{adbc} = 0 \tag{422}$$

$$R_{cbda} + R_{cabd} + R_{cdab} = 0 \tag{423}$$

$$R_{dcab} + R_{dabc} + R_{dbca} = 0 \tag{424}$$

$$R_{badc} + R_{bcad} + R_{bdca} = 0 \tag{425}$$

From (422), (423), (159) and (160), we obtain

$$R_{abcd} + R_{adbc} = R_{cdab} + R_{bcad} \tag{426}$$

Given this, notice that we can derive $R_{abcd} = R_{cdab}$, if we prove

$$R_{abcd} - R_{adbc} = R_{cdab} - R_{bcad} \tag{427}$$

We can prove this, by adding (424) and (425), and using (159) and (160). Thus, we conclude $R_{abcd} = R_{cdab}$.