The Gauss-Bonnet theorem for triangle on a sphere

In our earlier article "Curved space," we promised to obtain an expression for the area of a triangle on a sphere in terms of the sum of its angles. For simplicity, we will consider a sphere with area 720. You will soon see why the calculation is simple in this case. For spheres with other area, you can simply multiply the area of triangle in this case by appropriate factors.

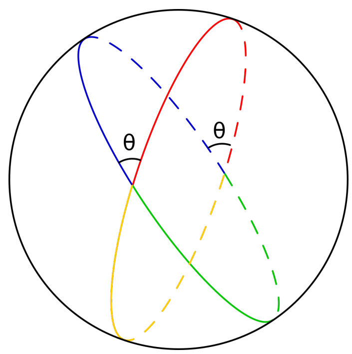

Figure 1: Two great circles dividing the sphere in four diangles. The diangle delimited by the blue and red curves has both angles θ.

To begin with, let's first consider the area of "diangle." A diangle is an object that has two sides and two vertices, just like a triangle is an object that has three sides and three vertices, and a quadrangle four sides and four vertices. Even though diangles cannot exist on a flat space, it can exist on a sphere. See Fig. 1. You see a sphere with two great circles. These two great circles divide the sphere into four diangles: a diangle with the blue side and the red side, a diangle with the red side and the green side, a diangle with the green side and the yellow side, and a diangle with the yellow side and the blue side. Here, it is very easy to see that the diangle with the blue side and the red side is congruent to (i.e., having the same shape and the same size) the diangle with the green side and the yellow side. Likewise, the diangle with the red side and the green side is congruent to the diangle with the blue side and the yellow side. Notice also that the two angles of a diangle are always the same. For example, in the diangle with the blue side and the red side, the two angles are both θ as marked in the figure.

What are the areas of these triangles? First of all, remember that a great circle divides a sphere into two equal half-spheres. As 360 is the area of each half-sphere, we have

$$\text{The area of blue red diangle} + \text{the area of red green diangle} = 360 \tag{1}$$

As each angle of the blue-red diangle is θ, each angle of the red-green diangle is given by $180^\circ - \theta$. In other words, we have

$$\text{The angle of blue red diangle} + \text{the angle of red green diangle} = 180^\circ \tag{2}$$
$$\theta + (180^\circ - \theta) = 180 \tag{3}$$

Now, observe that the blue-red diangle can be sliced into the same θ diangles with each angle 1°; we can simply divide the angle θ of the blue-red diangle into θ copies of 1°. Similarly, the red-green diangle can be sliced into the same $180 - \theta$ diangles with each angle 1°. Therefore, the blue-red diangle has θ times the area of the diangle with angle 1°, and the red-green diangle has $180 - \theta$ times the area of the diangle with angle 1°. Therefore, the sum of the areas of these two diangles is 180 times the area of the diangle with angle 1°. If we now recall that this sum is 360, we conclude that the diangle with angle 1° has the area 2. Therefore, we conclude the blue-red diangle has area 2θ and the red-green diangle $2(180 - \theta)$.

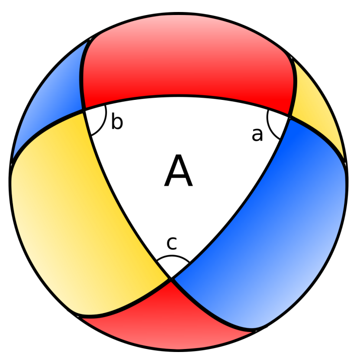

Figure 2: Three great circles forming the triangle A, with angles a, b and c. The two blue regions correspond to two different diangles that have the same angle b. These two diangles continue behind the sphere and meet at one of the vertices of a triangle there, just as they meet at one of the vertices of a triangle on the front side. (This vertex is marked with the angle b.) Similar statements can be made about the red and yellow regions, corresponding to the angles c and a, respectively

Now, we are fully ready to attack the problem. See Fig. 2. You see a sphere with three great circles, which form a triangle with angles a, b, c. The triangle has area A. For convenience, I didn't draw the other side of the sphere. Therefore, we only see the half of the sphere now. Nevertheless, it doesn't matter at all. If you look this sphere from the other side, you will see exactly the same picture as in Fig. 2, but with the direction reversed.

Notice now that we see part of two diangles with angle a in Fig. 2. As each of these two diangles has area $2a$, their total area is $4a$. Of course, we do not know the area of each part visible in Fig. 2, but if we add them that should be the half of the area of these two diangles, as we can see the other half from the opposite side. Therefore, the total area of the parts you see of these two diangles is half of $4a$, which is $2a$. This implies that the total area of the yellow regions is $2a - A$. Similarly, the total area of the blue regions is $2b - A$, and the one of the red is $2c - A$. As the area of half-sphere we see is 360, we must have the following relation:

$$\text{The yellow area} + \text{the blue area} + \text{the red area} + A = 360 \tag{4}$$

$$(2a - A) + (2b - A) + (2c - A) + A = 360 \tag{5}$$

$$2a + 2b + 2c - 2A = 360 \tag{6}$$

Another way of saying (6) is that the half of the two diangles with angle a, the half of the two diangles with angle b and the half of the two diangles c cover the area A three times, when we sum their areas to calculate the area of half-sphere. Notice that we should have included the area A only once. Thus, we have included it twice more. Thus, to caculate the area of half-sphere, we have to subtract $2A$ from the sum of all the areas of these diangles. In other words,

$$2a + 2b + 2c - 2A = 360 \tag{7}$$

which is exactly (6). In conclusion, we have

$$A = (a + b + c) - 180 \tag{8}$$

In other words, if we denote the sum of the three angles of a triangle by S, we have

$$A = S - 180 \tag{9}$$

Here, we can see that if the area of triangle A is very small, the sum of the angle is very close to 180°. As mentioned in an earlier article, this means that, when the triangle is very small, the effect of curvature is negligible and the triangle can be practically regarded as lying on a flat surface.

Problem 1. Remember that the above expression is valid only for the sphere with area 720. How will the above relation change, if the sphere has double the area, i.e., 1440? Or, more generally, how will the above relation change for sphere with the area $4\pi r^2$? (i.e., a sphere with radius r)

Problem 2. In the earlier article "Similarity of triangle," we learned that two triangles (on a flat plane) are similar if two angles of one triangle are the same as two angles of the other triangle. Is this criterion of similarity also valid for triangles on a sphere? Explain your reasoning.

Problem 3. Explain why two triangles on a sphere are congruent, if the three angles of one triangle are the same as the ones of the other. Explain also why this criterion for congruency doesn't work for two triangles on a flat plane.

Problem 4. Explain why it is impossible that two triangles on a sphere is not congruent, but similar.

Final comment. The relation that we found in this article can be derived from the Gauss-Bonnet theorem introduced in an earlier article. Therefore, the title of this article. In this article, we considered the case of surface with constant positive curvature i.e., a sphere. In case of a surface with constant negative curvature, the area of triangle is proportional to $180 - S$. In other words, the bigger the triangle the smaller the sum of its three angles. Or, the closer the sum of the angles is to $180°$, the smaller the triangle.

Problem 5. Explain why there is a maximum limit for the area of triangle on a surface with consant negative curvature.

Summary

- The area of a triangle on a sphere is proportional to $S - 180°$, where S is the sum of its three angles.
- Two triangles on a sphere are congruent, if the three angles of one triangle are the same as the three angles of the other triangle.