Gravitational field on the surface of the Earth

In the last article, we witnessed the power of Gauss's law. We will witness another one in this article. Suppose we have spherically symmetric electric charges distributed on a ball with radius R. It means that the charge density depends only on the distance from the center, and not on the direction from the center. See Fig. 1. Let me give you another example for spherical symmetry. The distribution of matter inside the Earth is roughly spherical symmetric, as the mass density and the phase of matter inside the Earth only depends on the distance to the center. (However, it doesn't have the exact spherical symmetry as the shape of the Earth is not perfect sphere.) We have solid within 1216 km from the center of the Earth, and liquid between 1216 km and 3486 km from the center of Earth. All that matters is the distance from the center. Given this, what would be the electric field at the point distance r from the center? For a Gaussian surface, we can take a sphere with radius r centered around the origin of the ball as in the figure. Also, we know that the electric field must be in the radial direction because of symmetry. (Remember that the charge is distributed spherically symmetric.) Therefore, if the charge inside the Gaussian surface is say $q(r)$, the electric field at the position r will be proportional to $q(r)/(4\pi r^2)$. In other words, the electric field at the position r will be as if all the charge $q(r)$ were at the center. As explained in the last article, the charge outside this Gaussian surface doesn't contribute to the electric field at the Gaussian surface in case the charge is distributed in the spherically symmetric manner. Similarly, it is also easy to see that the electric field at $r = R$ will be as if all the total charge in the ball were at the center.

Now, let's slightly change the topic. In "Conic sections and Newton's law of gravity," I briefly mentioned that an apple falls to the ground because of Newton's universal gravitational force which is inversely proportional to

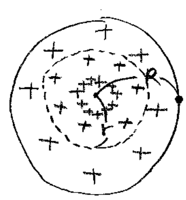

Figure 1: spherically symmetric charges on a ball

the square of distance, and is experienced by every massive object. Actually, in this case, the massive object is the Earth. In "Free fall," I explained that this gravitational force must be proportional to the mass of the object (i.e. apple), as we observe that the free fall acceleration is independent of the mass of the object. In other words, the gravitational force is proportional to the mass of the object attracted. To find out heuristically how the gravitational force is related to the mass of the object that attracts, consider this example. Let's say that the mass of three objects A, B and C are each 2 kg, and they are to be located at the same position (this would be ideal, but in reality they would be located at close positions). Also, let's say that there is an object D which is 1 km away from the three objects. Let's say that A attracts D with 10 Newtons. Then, B would attract D with 10 Newtons and C as well, as they have the same mass. Then, if we count A, B, C as one object called "ABC," then it attracts D with 30 Newtons, because 10 Newtons times 3 is 30 Newtons. We can re-interpret this as this way. As ABC has the triple mass of A, the gravitational force is tripled. Thus, the gravitational force is proportional to the mass of the object that attracts. Summarizing, our conclusion is that the gravitational force is proportional to the mass of the object attracted (let's call it m), and the mass of the object attracts (let's call it M) and inversely proportional to the square of the distance between them (let's call it r). In other words, Newton's universal gravitational force is given by

$$F = G\frac{Mm}{r^2} \tag{1}$$

where G is a proportionality constant called "Newton's constant" and measured to be about $6.674 \times 10^{-11}\mathrm{m}^3 \cdot \mathrm{kg/s}^2$

Let's apply this formula to a apple on a tree. What is the gravitational force? m is the mass of the apple, and M is the mass of the Earth, but what should r be? The size of the Earth is so large that the distance from a part of the Earth to the apple is very different from another part of the Earth to the apple. We cannot single out a number for the distance; it can be few meters to the diameter of the Earth.

Now comes the power of Gauss's law. As mentioned earlier, we can apply Gauss's law to the gravitational field because it also follows inverse square law like the electric field. Let's apply Gauss's law to the gravitational field of the Earth. As mentioned, the Earth is a sphere to a very good approximation, and its mass is distributed almost spherically symmetrically. By following the same logic as in our earlier example in this article, the gravitational field at radius r is given as if all the mass inside the radius r were at the center. Similarly, when r is the radius of the Earth R (i.e. at the surface of the Earth) the gravitational field will be given as if all the mass of the Earth were at the center of the Earth. Therefore, the gravitational force on the apple is given by

$$F = G\frac{Mm}{R^2} \tag{2}$$

as the distance from the center of the Earth to the apple is the radius of the Earth. As the above expression is $F = mg$, we get also a nice relation

$$g = \frac{GM}{R^2} \tag{3}$$

Problem 1. Let's say that a planet is a sphere with radius b, and the density is ρ_1 when $0 < r < a$ and ρ_2 when $a < r < b$ (r is the distance to the center). What is g (i.e. the gravitational force per mass) when $0 < r < a$, $a < r < b$ and $r > b$? (Hint[1])

Summary

- If the mass in a planet is distributed in such a way that there is spherical symmetry (i.e., the density depends only on the distance from the center) then the gravitational field at a point distance r from the center of the planet can be calculated as if all the mass inside the Gaussian sphere with radius r is located at the origin; the total gravitational field due to the mass outside the Gaussian sphere is zero.

[1] First calculate the mass inside the radius r.