The Gibbs factor

When we derived the Bose-Einstein distribution and the Fermi-Dirac distribution in our earlier article, we defined the chemical potential by relating it with the Lagrange multiplier. In this article, we will define the chemical potential slightly differently, and derive the Bose-Einstein distribution and the Fermi-Dirac distribution using this new chemical potential, and obtain the same results for both ditribution as the ones in our earlier article, which justifies our new definition of the chemical potential.

Recall that in our earlier article "Boltzmann factor" we considered a heat reservoir which can exchange heat with the small system we are considering. Now, we will assume a heat reservoir that can exchange not only heat, but also particles. Let's say that the total energy and the total number of particles in the reservoir and system are U_0 and N_0. Suppose now, the energy and the number of particles in the system are E and N. Then, the energy in the reservoir is $U_0 - E$ and the number of particles in the reservoir is $N - N_0$. We will also assume that the number of possible states for such a system is 1, since we are considering a particular state. Then, the total entropy is given by the sum of the entropy of the reservoir and the entropy of system as follows:

$$S_{\text{total}} = S(U_0 - E, N_0 - N) + k \ln 1 \tag{1}$$

Now, I will define the chemical potential as follows:

$$\mu \equiv -T \frac{\partial S}{\partial N} \tag{2}$$

Then, (1) becomes

$$S_{\text{total}} = S(U_0, N_0) - E\frac{\partial S}{\partial U} - N\frac{\partial S}{\partial N} = S(U_0, N_0) - \frac{E}{T} + N\frac{\mu}{T} \tag{3}$$

Thus, just as in our earlier article "Boltzmann factor" we obtain so-called "Gibbs factor" for the probability ratio:

$$\frac{P(E_1, N_1)}{P(E_2, N_2)} = \frac{\exp[(N_1\mu - E_1)/kT]}{\exp[(N_2\mu - E_2)/kT]} \tag{4}$$

Just like the partition function, we can define the grand partition function as follows:

$$\mathcal{Z} = \sum_{N=0}^{\infty} \sum_{s(N)} \exp[\beta(N\mu - E_{s(N)})] \tag{5}$$

where $s(N)$ denotes the various possible states N particles can have.

Let's now consider the Fermi-Dirac distribution. There are only two possible states. $N = 0$ and $N = 1$. If the energy of the fermion is ϵ, the $N = 1$ state has ϵ energy more than the $N = 0$ state. Thus,

$$\frac{P(0,0)}{P(\epsilon,1)} = \exp[(\mu - \epsilon)/kT] \tag{6}$$

As we have $P(0,0) + P(\epsilon,1) = 1$, we conclude

$$P(\epsilon,1) = \frac{1}{e^{(\epsilon-\mu)/kT} + 1} \tag{7}$$

The number of particle for $E = 0, N = 0$ is 0 and the number of particle for $E = \epsilon, N = 1$ is 1. So, the expectation value of the number of particle is given by

$$0 \times P(0,0) + 1 \times P(\epsilon,1) = \frac{1}{e^{(\epsilon-\mu)/kT} + 1} \tag{8}$$

This is precisely the Fermi-Dirac distribution.

Problem 1. Similarly, obtain the Bose-Einstein distribution! (Hint[1])

Problem 2. Show that

$$\langle N\mu - E\rangle = \frac{\partial \ln \mathcal{Z}}{\partial \beta} \tag{9}$$

Summary

- The chemical potential is defined by

$$\frac{\mu}{T} \equiv -\frac{\partial S}{\partial N}$$

- The Gibbs factor is given by

$$\exp[\beta(N\mu - E)]$$

  and the grand partition function is the sum of the Gibbs factor for all possible states.

[1]Recall how we obtained Planck's law of blackbody radiation. Use also $E_{s(N)} = N\epsilon$. Then, what we want to calculate is

$$\frac{1}{\mathcal{Z}} \sum_{N=0}^{\infty} N \exp[\beta(N\mu - N\epsilon)]$$