The Hamiltonian formulation of classical mechanics

Recall the Euler-Lagrange equation in our previous article. We had:

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}^i}) = 0 \tag{1}$$

If we define the "conjugate momentum" of q^i as follows:

$$p_i \equiv \frac{\partial L}{\partial \dot{q}^i} \tag{2}$$

we can write the Euler-Lagrange equation as:

$$\dot{p}_i = \frac{\partial L}{\partial q^i} \tag{3}$$

This allows us to express the small variation of Lagrangian as follows:

$$\begin{aligned} \delta L &= \sum_i (\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i) \\ &= \sum_i \dot{p}_i \delta q^i + p_i \delta \dot{q}^i \end{aligned} \tag{4}$$

Stepping further, we get:

$$\begin{aligned} \delta L &= \sum_i \dot{p}_i \delta q^i + \delta(p_i \dot{q}^i) - \dot{q}^i \delta p_i \\ \delta(\sum_i p_i \dot{q}^i - L) &= \sum_i -\dot{p}_i \delta q^i + \dot{q}^i \delta p_i \end{aligned} \tag{5}$$

We now define

$$H \equiv \sum_i p_i \dot{q}^i - L \tag{6}$$

and call it "Hamiltonian." It turns out that this coincides with energy, as we you will see in our later examples in this article. Furthermore, by solving Problem 1, you will be convinced that Hamiltonian is energy. At this point, I want to caution the readers that H must be re-expressed solely in terms of p_is and q^is by eliminating $\dot{q}^i$. In other words, $\dot{q}^i$ must be absent in H.

From (5) and (6) , we have:

$$\delta H = \sum_i -\dot{p}_i \delta q^i + \dot{q}^i \delta p_i \tag{7}$$

which implies

$$\frac{\partial H}{\partial q^i} = -\dot{p}_i, \qquad \frac{\partial H}{\partial p_i} = \dot{q}^i \tag{8}$$

These are called Hamilton's equations.

Let me give you an example. In 3-dimensional Cartesian coordinate, we have:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \tag{9}$$

Here, we will set:

$$q^1 = x, \qquad q^2 = y, \qquad q^3 = z \tag{10}$$

Then, from (2), we have:

$$p_1 = m\dot{q}^1, \qquad p_2 = m\dot{q}^2, \qquad p_3 = m\dot{q}^3 \tag{11}$$

In other words:

$$p_i = m\dot{q}^i \tag{12}$$

Now, we can re-express Lagrangian in terms of ps and qs, instead of $\dot{q}$s and qs as follows:

$$L = \sum_i \frac{(p_i)^2}{2m} - V(q_i) \tag{13}$$

Plugging this into (6), we get:

$$\begin{aligned} H &= \sum_i p_i(\frac{p_i}{m}) - L \\ &= \sum_i \frac{(p_i)^2}{2m} + V(q^i) \end{aligned} \tag{14}$$

Then, Hamilton's equations become:

$$\frac{\partial H}{\partial p_i} = \frac{p_i}{m} = \dot{q}^i \tag{15}$$

$$p_i = m\dot{q}^i \tag{16}$$

$$\frac{\partial H}{\partial q^i} = \frac{\partial V}{\partial q^i} = -\dot{p}_i = -m\ddot{q}^i \tag{17}$$

$$m\ddot{q}^i = -\frac{\partial V}{\partial q^i} \tag{18}$$

So, we recover Newton's equations. Notice also that (14) is the usual energy, i.e., the sum of kinetic energy and the potential energy. Of course, Hamilton's equations are always valid for general cases as well, not just when the position and momentum of particle are expressed by the Cartesian coordinate.

Problem 1. Energy is conserved. In other words, if the Hamiltonian is energy, it also must be conserved. In this problem, you will show that Hamiltonian is conserved. Assume that Hamiltonian H only depends on p_i and q^i. In other words, $H = H(p_i, q^i)$. Then, by using chain rule and (8), show that

$$\frac{dH}{dt} = 0 \tag{19}$$

Problem 2. In our earlier article "Central force problem solution in terms of Lagrangian mechanics," we obtained the equation of motion for the case in which the Lagrangian was given as follows:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{20}$$

First, obtain the Hamiltonian in terms of $r, \theta, \dot{r}$ and $\dot{\theta}$, sin such a case and check that it coincides with the usual energy. (i.e. the sum of the kinetic energy and the potential energy) Then, re-express the Hamiltonian in terms of r, θ, p_r and p_θ, and obtain the equation of motion using Hamilton's equations and check that they coincide with the earlier ones obtained using Lagrangian formulation of classical mechanics.

Let me conclude this article with a comment. The process in which we obtained Hamiltonian from Lagrangian may seem like a magic but it is just an example of what is called "Legendre transformation." The Legendre transformation of function $f(x, y)$ is given by

$$g(p, y) = f - px \tag{21}$$

where $p = \frac{\partial f}{\partial x}$. In our case we had $x = \dot{q}$, $y = q$, $f = L$, and $g = -H$. Besides the Hamiltonian mechanics, Legendre transformation has applications in thermodynamics and quantum field theory.

Summary

- The conjugate momentum of q^i is given by

$$p_i \equiv \frac{\partial L}{\partial \dot{q}^i}$$

- The Hamiltonian is then given by

$$H = \sum_i p_i \dot{q}^i - L$$

- Hamilton's equations are given by

$$\frac{\partial H}{\partial q^i} = -\dot{p}_i, \qquad \frac{\partial H}{\partial p_i} = \dot{q}^i$$

- The Hamiltonian corresponds to the energy (i.e. the sum of kinetic energy and potential energy).