Integration and the fundamental theorem of calculus

1 Motivation for integration

Suppose you are initially (i.e. $t = 0$) at position $x = 0$ m, then, move 10 m/s in positive x-direction for 3.5 seconds then 20 m/s in positive x-direction for 4.5 seconds. Then, where are you now? It is very easy. We have:

$$0\text{m} + 10\text{m/s} \times 3.5\text{s} + 20\text{m/s} \times 4.5\text{s} = 125\text{m} \tag{1}$$

If you draw time versus velocity graph as in Fig. 1, it is easy to see that the distance traveled is equal to the area enclosed by the graph, the axis of time, the lines $t = 0$ and $t = 8$.

To gain some insight, let's consider another problem. Suppose you are initially (i.e. $t = 0$) at position $x = 0$. Then, move 10 m/s in positive x-direction for 3.5 seconds then 20 m/s in negative x-direction for 4.5 seconds. Then, where are you now? It is very easy. We have:

$$0\text{m} + 10\text{m/s} \times 3.5\text{s} - 20\text{m/s} \times 4.5\text{s} = -55\text{m} \tag{2}$$

If you draw time versus velocity graph as in Fig. 2, it is easy to see that the distance traveled is equal to the area enclosed by the graph, the axis of time, the lines $t = 0$ and $t = 8$ as before. However, the position is different. When $3.5 \leq t \leq 8$, you actually moved left-ward, so you have to deduct the distance traveled to obtain the position. This is what I mean by negative sign in "-90 m" in the figure. Therefore, the position change is given by the sum of "signed" area of the graph (i.e. "$-$" sign in "-90 m"): 35 m$-$90 m=$-$55 m. If you add the initial position "0 m", you get the position -55 m.

Now, we can explain what integration is. Integration of a function is the sum of "signed" area bounded by x-axis, the function, and an x-interval. From our examples, it is clear that

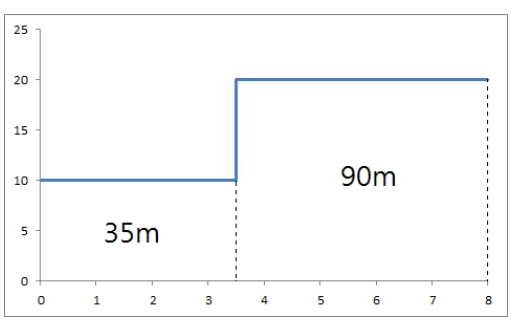


Figure 1: First case

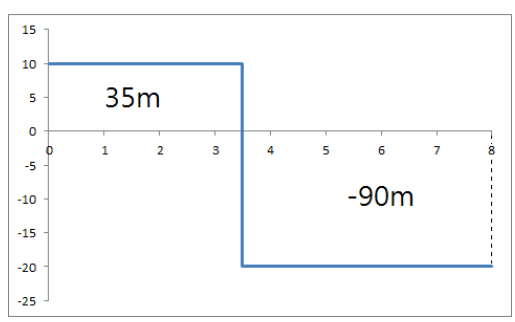


Figure 2: Second case

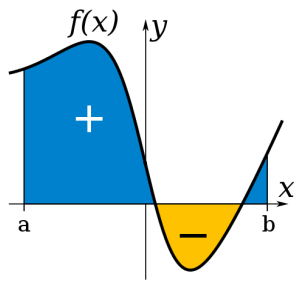


Figure 3: Integration of $f(x)$

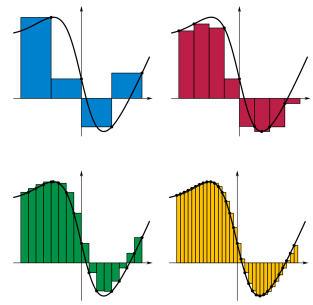


Figure 4: Integration as a limit of sums

integration of velocity is position. As we know that velocity is derivative of position, it is clear that integration is the reverse process of differentiation.

Given this, let me explain how we denote integration. If velocity is given by $v(t)$, then the position change Δx between t_1 and t_2 is equal to the following:

$$\Delta x = \int_{t_1}^{t_2} v(t)dt \tag{3}$$

In our first example, we had:

$$125 = \int_0^8 v(t)dt \tag{4}$$

In our second example, we had:

$$-55 = \int_0^8 v(t)dt \tag{5}$$

In general, the function integrated, called "integrand," changes continuously unlike in our examples (in our examples it changed only once at t=3.5 s) and the calculation of the area could be more complicated. See Fig.3. We see that the sum of the signed area is the integration of the function $f(x)$. To calculate the area approximately in cases such as this, we chop up the x-axis into many small intervals and calculate the value of the function at one of the two endpoints of the interval and multiply it by the length of the interval. This will give each of the rectangles in Fig.4. And if you sum them up, you will get the total area. There are four figures in Fig. 4. The first one has only four intervals. Very crude. It doesn't approximate the area quite well. However, the second one has more intervals with less width. It approximates the area better. From the third, and the fourth ones, you easily see that the more intervals and the less width, the better approximation of the area. Actually, integration is defined by the sum of the signed area of these rectangles in the limit that the width of each interval goes to zero. This explains the notation for integration. The sum of each rectangles can be roughly expressed as follows:

$$\sum_i f(x_i)\Delta x \tag{6}$$

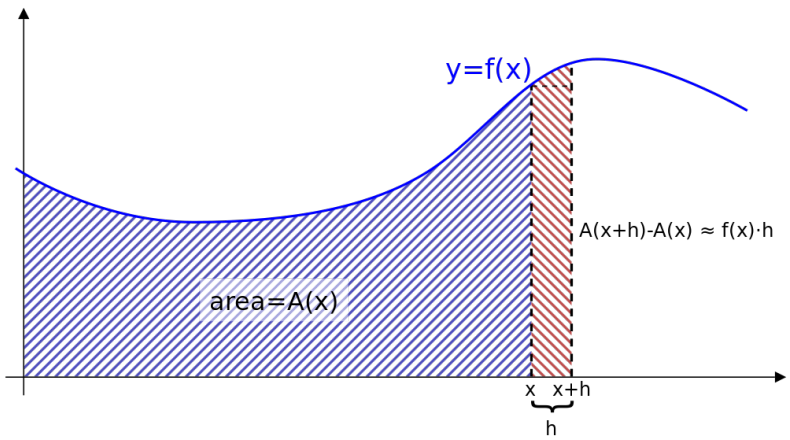


Figure 5: Fundamental theorem of calculus

where i denotes each interval. Now, the sum becomes the exact value of the area when Δx goes to zero. Therefore, in such a case, we replace Δx by dx, and the summation symbol $\sum$ by integration symbol $\int$. Therefore, we have:

$$\int_a^b f(x)dx \tag{7}$$

2 Definite integral and indefinite integral

Now, let's go back to our velocity-position interpretation of integration. If we denote the position at time t by $x(t)$, from (3) we certainly have:

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} v(t)dt \tag{8}$$

Therefore, it turns out that it is convenient to define an integral without range (called "indefinite integral") as follows:

$$x(t) = \int v(t)dt \tag{9}$$

Once we obtain an explicit function $x(t)$, we can calculate the value for an integral with range (called "definite integral") such as (8) by plugging the values for t_2 and t_1 and subtract them. All will be clear once we give some examples later in the article.

3 Fundamental theorem of calculus

Earlier in the article, we explained that the reverse process of differentiation is integration, by using the relation between velocity and position. However, this can be shown using the area picture of the integration as well. This is called "fundamental theorem of calculus."

Suppose you integrate a function $f(x)$ and obtained $A(x)$. In other words,

$$A(x) = \int f(x)dx \tag{10}$$

See Fig. 5. Given this, let's calculate the derivative of $A(x)$.

$$\frac{dA}{dx} = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} \tag{11}$$

Now, from the figure the following is clear for very small h:

$$A(x+h) - A(x) \approx f(x)h \tag{12}$$

Plugging this back to (11) we obtain:

$$\frac{dA}{dx} = f(x) \tag{13}$$

as advertised. In fact, we can prove it somewhat more rigorously. In our case in which $f(x+h) \geq f(x)$, from the figure, the following is clear:

$$f(x)h \leq A(x+h) - A(x) \leq f(x+h)h \tag{14}$$

which implies

$$f(x) \leq \frac{A(x+h) - A(x)}{h} \leq f(x+h) \tag{15}$$

$$\lim_{h \to 0} f(x) \leq \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} \leq \lim_{h \to 0} f(x+h) \tag{16}$$

$$f(x) \leq \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} \leq f(x) \tag{17}$$

Using (11), we conclude (13). Case in which $f(x+h) \leq f(x)$ can be proven similarly. In either case, the derivative of $A(x)$ falls between the value of $f(x)$ and $f(x+h)$.

4 Integration of polynomials

Let's calculate the following quantity:

$$\int_2^4 (x+3)dx \tag{18}$$

To this end, we have to find a function that becomes $x+3$ upon differentiation. We know $(x^2)' = 2x$. Therefore $(\frac{1}{2}x^2)' = x$. We also know $(3x)' = 3$, and $(C)' = 0$ for any constant C. Therefore, we have

$$(\frac{1}{2}x^2 + 3x + C)' = 2x + 3 \tag{19}$$

In other words,

$$\int (x+3)dx = \frac{1}{2}x^2 + 3x + C \tag{20}$$

Notice that C should be always present in the result of any integration, as its derivative is zero. Such a C is called "integration constant."

Using our earlier explanation between definite integral and indefinite integral, we conclude:

$$\int_2^4 (x+3)dx = \left(\frac{1}{2}\cdot 4^2 + 3\cdot 4 + C\right) - (\frac{1}{2}\cdot 2^2 + 3\cdot 2 + C) = 12 \tag{21}$$

where we see that C, the integration constant, is always cancelled for definite integral. Some books use the following notation for definite integral:

$$\int_2^4 (x+3)dx = \frac{1}{2}x^2 + 3x\Big|_2^4 = 12 \tag{22}$$

In other words, you plug in 4 to x to calculate $x^2/2 + 3x$ and then, you subtract from it the value you get when you plug in 2 to x.

Generally speaking, for $n \neq -1$, it is easy to check the following:

$$\left(\frac{x^{n+1}}{n+1} + C\right)' = x^n \tag{23}$$

Therefore, we conclude:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \tag{24}$$

We emphasize that this is not valid when $n = -1$ as denominator becomes zero. In other words, we cannot use the above formula to integrate $1/x$. However, remember our earlier article "Exponential function and natural log." There, we have seen

$$(\ln x)' = \frac{1}{x} \tag{25}$$

Therefore, we conclude:

$$\int \frac{1}{x} dx = \ln x + C \tag{26}$$

In any case, (24) and (26) are very useful. An example:

$$\begin{aligned} \int (2x^3 + 4x - 5 + \frac{2}{x} - \frac{1}{x^2})dx &= \frac{2x^4}{3+1} + \frac{4x^2}{1+1} - 5x + 2\ln x - \frac{x^{-1}}{-2+1} + C \\ &= \frac{1}{2}x^4 + 2x^2 - 5x + 2\ln x + \frac{1}{x} + C \end{aligned} \tag{27}$$

5 Integration of trigonometric function and exponential function

We have:

$$(\sin(ax))' = a\cos(ax), \qquad (\cos(ax))' = -a\sin(ax), \qquad (e^{ax})' = ae^{ax} \tag{28}$$

Therefore, one can check:

$$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + C \tag{29}$$

$$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + C \tag{30}$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C \tag{31}$$

6 The equation of motion for a constant acceleration

Now, some applications. Let's consider an object whose acceleration is constant, given by a_0. What would be its position x?

We have $a_0 = \frac{dv}{dt}$ which implies:

$$v = \int a_0 dt = a_o t + C_1 \tag{32}$$

We also have $v = \frac{dx}{dt}$ which implies:

$$x = \int v dt = \int (a_0 t + C_1)dt = \frac{1}{2}a_0 t^2 + C_1 t + C_2 \tag{33}$$

for some constants C_1 and C_2. By plugging $t = 0$ into (32), we see that C_1 is the initial velocity (i.e. when $t = 0$). By plugging $t = 0$ into (33), we see that C_2 is the initial position.

7 Remark

Final comment. At first glance, it may seem that we can integrate any function easily, as much as we can differentiate any function easily using Leibniz rule and/or quotient rule. However, it is not so. Consider the following example:

$$\int \frac{dx}{x\sqrt{a^2 + x^2}} = ? \tag{34}$$

The integrand (i.e. the function being integrated) is not in the form of polynomial functions, trigonometric functions or exponential functions, which we already know how to integrate. Even though we will learn a couple of more tricks to integrate in later articles, in most cases the integration is still not easy; you sometimes need a good guess. But, this guess is more often than not hard. To show you how hard it is let me state the answer to the above integration:

$$\int \frac{dx}{x\sqrt{a^2 + x^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 + x^2}}{x}\right| + C \tag{35}$$

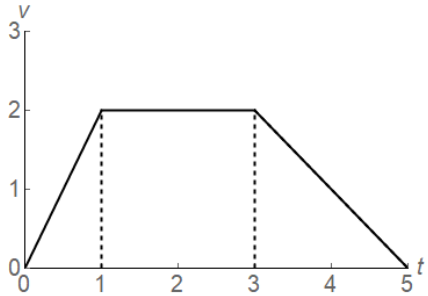
On the other hand, it is much easier to check that the above answer is correct, because as mentioned differentiation is easy even though it can be complicated.

Sometimes, the situation is worse. There can be no closed form expressions for integrations of some functions. (By closed form expressions, I mean expressions that can be expressed by usual functions such as trigonometric function, exponential function and so on.)

Anyhow, to aid integration, scientists used to use thick dictionaries where integrands of many functions and their integration are listed. However, as much as no one uses physical

copies of yellow pages and white pages any more, these thick dictionaries are now replaced by computer programs such as "Mathematica," first developed by ex-physicist Wolfram.

Problem 1. Suppose a car is moving along a straight line, and its velocity $v(t)$ is given by the following graph. Find the distance the car traveled for $0 \leq t \leq 5$.



Problem 2.

$$\int \left(\sqrt{2x} + 1 + \cos(3x)\right) dx = ?, \qquad \int_4^9 \frac{1}{\sqrt{x}} dx = ? \tag{36}$$

Problem 3. (Hint[1])

$$\int \sin^2(kx) dx = ? \tag{37}$$

This problem is useful in our later article "infinite potential well."

Problem 4.

$$\int 4e^{-2x} dx = ? \tag{38}$$

Problem 5. Show the following.

$$\int \frac{dx}{x+2} = \ln(x+2) + C, \qquad \int \frac{dx}{3x+5} = \frac{1}{3}\ln(3x+5) + C \tag{39}$$

Problem 6.

$$\int \frac{dx}{(x+3)^2} = ?, \qquad \int (x+m)^n dx = ? \tag{40}$$

Problem 7. (Hint[2])

$$\int \frac{x^2}{x+2} dx = ?, \qquad \int dx \left(\frac{1}{x} - \frac{1}{x+1} + \frac{1}{(x+1)^2}\right) = ? \tag{41}$$

Problem 8. (Hint[3])

$$\int \frac{dx}{x(x+2)} = ? \tag{42}$$

Problem 9. (Hint[4])

$$\int \frac{(x+\frac{1}{2})dx}{x(x+1)^2} = ? \tag{43}$$

This problem is useful for quantum field theory later.

[1] Use $\sin^2 x = (1 - \cos 2x)/2$.

[2] $\frac{x^2}{x+2} = x - 2 + \frac{4}{x+2}$

[3] $\frac{1}{x(x+2)} = \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+2}\right)$

[4] Re-express the integrand as $\frac{a}{x} + \frac{b}{x+1} + \frac{c}{(x+1)^2}$.

Summary

- Integration of a function is the sum of "signed" area bounded by x-axis, the function, and an x-interval.
- Integration of velocity is position.
- Integration is the reverse process of differentiation.
- Integration can be understood as the following limit

$$\lim_{\Delta x \to 0} \sum_i f(x_i)\Delta x = \int_a^b f(x)dx$$

  where a and b denote the range of the integration.
- Definite integral has a range, and indefinite integral doesn't have a range.
- Fundamental theorem of calculus says that

$$A(x) = \int f(x)dx$$

  implies $\dfrac{dA(x)}{dx} = f(x)$.
- The result of an indefinite integral always involves the integration constant. Nevertheless, the integration constant is cancelled out in the definite integral.
- $$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \qquad \int \frac{1}{x} dx = \ln x + C$$
- $$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + C, \qquad \int \sin(ax)dx = -\frac{1}{a}\cos(ax) + C$$
$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C$$

(Figure 3 is from `http://en.wikipedia.org/wiki/File:Integral_example.svg`. Figure 4 is adopted from `http://en.wikipedia.org/wiki/File:Riemann_sum_convergence.png`. Figure 5 is from `http://en.wikipedia.org/wiki/File:FTC_geometric2.png`)