Kinetic energy and Potential energy in one dimension

In this article, we will consider the case in which an object can move only along one-dimensional straight line. A general three-dimensional case will be considered in a later article.

Suppose the force exerted on an object depends only on its position. In other words, $F = F(x)$. Then, from Newton's second law we have:

$$F(x) = m\frac{dv_x}{dt} \tag{1}$$

where $v_x = dx/dt$. Now, let's multiply dx on both hand-sides, and integrate. Then, we get:

$$\int F(x)dx = \int m\frac{dv_x}{dt}dx = \int mv_x dv_x = \frac{1}{2}mv_x^2 + \text{constant} \tag{2}$$

$$0 = \frac{1}{2}mv_x^2 + (-\int F(x)dx) + \text{constant} \tag{3}$$

Given this, let's define the kinetic energy as follows

$$K = \frac{1}{2}mv_x^2 \tag{4}$$

and the potential energy $U(x)$ as follows:

$$U(x) = -\int F(x)dx \tag{5}$$

and the total energy E as follows:

$$E = K + U \tag{6}$$

Then, (3) implies that the total energy E is constant. In other words, we say that the total energy is conserved. Now, an example. Suppose the potential energy is given as in Fig. 1. What would be the magnitude and the direction of the force, given a position? From (5), we see the following:

$$F = -\frac{\partial U}{\partial x} \tag{7}$$

Therefore, when $x < a$ or $x > b$ the force is rightward, when $a < x < b$ the force is leftward. When $x = a$ or $x = b$, there will be no force. Also, as (7) shows, the steeper the potential energy the bigger the force.

Therefore, you can almost think the potential energy as "terrain." Suppose you place a little ball on the terrain like Fig. 1. If you place the ball at a place $x < a$ or $x > b$ the ball will feel a force rightward. If you place the ball at a place $a < x < b$ it will feel a force leftward. Also, you can easily imagine that the force will be greater the steeper the terrain. All these pictures agree with our earlier analysis by (7).

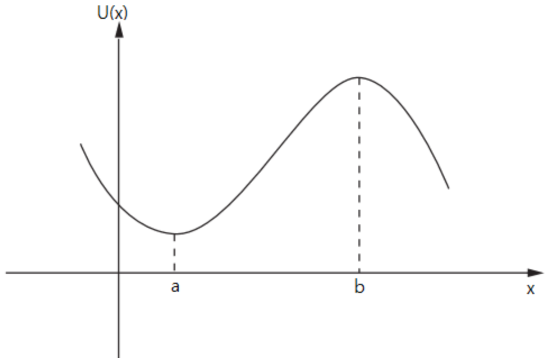

Figure 1: U(x)

When $x = a$ or $x = b$, no force is exerted. We call this case the ball is at "equilibrium." However, their natures are different in these two cases. Suppose you place a ball near $x = a$, then it will roll toward $x = a$ the lowest place, then it will oscillate around $x = a$. However, if you place a ball near $x = b$ then it will not oscillate back but go farther from the equilibrium position. In other words, if we place it left of the equilibrium position it will move leftwards. Similarly if we place it right of the equilibrium position it will move rightwards. Therefore, we call the first case, $x = a$, the stable equilibrium, while we call the second case $x = b$, the unstable equilibrium.

Now, let's say that we place an object at the place $x = 0$ with an initial velocity v_0. Then, will it go beyond the "hill" $x = b$? Let's answer this question. E, the total energy of the object is given as follows:

$$E = \frac{1}{2}mv_0^2 + U(0) \tag{8}$$

Then, what is $v(b)$, the velocity at $x = b$? At $x = b$ we have:

$$E = \frac{1}{2}mv(b)^2 + U(b) \tag{9}$$

$$v(b) = \sqrt{\frac{2(E - U(b))}{m}} \tag{10}$$

This holds when $E \geq U(b)$ is satisfied, since $v(b)$ is real. In such a condition, the object will go beyond the hill. However, if $E < U(b)$ the object will turn leftwards before it reaches $x = b$ as it doesn't have enough energy to climb up the $x = b$ hill. Where will it turn leftwards? At that position the velocity will be zero. If we call the position "x_0," we have:

$$E = \frac{1}{2}m \cdot 0^2 + U(x_0) = U(x_0) \tag{11}$$

So at the place in which x_0 satisfies the above formula, it will turn leftwards. This will be a position between a and b. This point is called "classical turning point" or simply "turning point." In our later article "Tunneling," we will see that particles can go beyond classical turning point according to quantum mechanics. The reason why "turning point" is sometimes called "classical turning point" is that this is the point at which particles turn according to classical mechanics.[1]

Problem 1. Find the potential energy when $F = -kx$. Does this configuration have equilibrium? If it does, is it stable or unstable? Describe qualitatively what the motion will look like.[2]

Problem 2. Find the stable equilibrium(s) or the unstable equilibrium(s) for the potential $U = \lambda(x^2 - a^2)^2$ where $\lambda > 0$ and $a > 0$. This potential is called Higgs potential, or Mexican hat potential. (Yes, this is the same Higgs whose particle is named after him. He came up with such a potential.)

Summary

- Energy conservation can be written as $E = K + U$
  where
  $$K = \frac{1}{2}mv^2, \qquad U(x) = -\int F(x)dx$$
  K is called kinetic energy and U is called potential energy.
- $F = -\dfrac{\partial U}{\partial x}$
- It is easy to understand how a particle moves in a given potential energy, if you regard the potential energy as the height in a terrain.
- Equilibriums are the points where $F = -\dfrac{\partial U}{\partial x}$ is zero.
- At a stable equilibrium, U is local minimum. At an instable equilibrium, U is local maximum.

[1]There is no such thing as "quantum turning point" because according to quantum mechanics particles do not follow simple trajectory as described in this article. What I mean will be clear when you read my articles on quantum mechanics.

[2]The answer is in the next article.