Kinetic energy and Potential energy in three dimensions, Line integrals and Gradient

Having introduced kinetic energy and potential energy in one dimension, we will introduce the analogous construction in three dimensions. As before, let's assume that the force only depends on position as follows:

$$F = F(\vec{s}) \tag{1}$$

where $\vec{s}$ denotes the position. Then, as before, we have:

$$m\frac{d\vec{v}}{dt} = F(\vec{s}) \tag{2}$$

$$\int m\frac{d\vec{v}}{dt} \cdot d\vec{s} = \int F(\vec{s}) \cdot d\vec{s} \tag{3}$$

$$\int m\vec{v} \cdot d\vec{v} = \int F(\vec{s}) \cdot d\vec{s} \tag{4}$$

$$\tag{5}$$

Therefore, we conclude:

$$\frac{1}{2}m\vec{v} \cdot \vec{v} - \int F(\vec{s}) \cdot d\vec{s} = \text{constant} \tag{6}$$

If we let the kinetic energy be defined as follows:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v} \cdot \vec{v} \tag{7}$$

and the potential energy be defined as follows:

$$U(\vec{r}) = -\int F(\vec{s}) \cdot d\vec{s} \tag{8}$$

as before, we see that the total energy $E = K + U$ is conserved (i.e. constant). The form of the integration on the right-hand side is called "Line Integrals." It involves integration of dot-products of vector field (i.e. $\vec{F}$) with small differential (i.e. $d\vec{s}$)

However, there is an additional subtlety as one goes from 1 dimensional case to 3 dimensional case. First, observe the following:

$$U(\vec{s}_2) - U(\vec{s}_1) = -\int_{\vec{s}=\vec{s}_1}^{\vec{s}=\vec{s}_2} F(\vec{s}) \cdot d\vec{s} \tag{9}$$

In general, the line integral on the right-hand side can depend on the path taken. See Fig.1. The integral along path A can differ from the integral along path B. However, the left-hand side only depends on the potential at the endpoints (i.e. at $\vec{r}_1$ and $\vec{r}_2$) and not

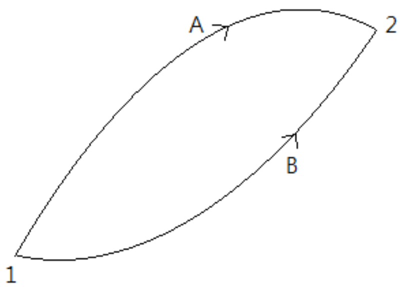

Figure 1: Path A and Path B

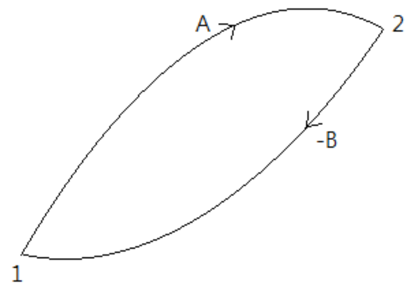

Figure 2: Closed path (Path A then Path -B)

on the intermediary points (i.e. the path). Therefore, by looking at the right-hand side, we can easily notice that for the potential energy to be well-defined, the line integral must be independent of the path taken once their endpoints are fixed. As we will see shortly, this condition is satisfied if the force $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ can be expressed in terms of the potential energy U as follows:

$$F_x = -\frac{\partial U}{\partial x}, \qquad F_y = -\frac{\partial U}{\partial y}, \qquad F_z = -\frac{\partial U}{\partial z} \tag{10}$$

If we plug this back to the right-hand side of (9), the equation is automatically satisfied. Let's check this. From chain-rule, we have:

$$\begin{aligned} dU &= \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz & (11) \\ &= -(-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z}) \cdot (dx, dy, dz) & (12) \\ &= -F(\vec{r}) \cdot d\vec{r} & (13) \end{aligned}$$

Integrating both-hand sides, we get (9) as advertised.

At this point, our earlier analogy with the terrain in 1-dimensional case could be helpful again in this multi-dimensional case, especially in 2-dimensional case. In this case, we have x, and y coordinates but no z coordinate, and U the potential energy can be seen as height as it was in 1-d case. Equation (10) and subsequent equations hold, but without the z part. In this case, if you climb from point 1 to point 2 the altitude you gained will be independent of the path you have taken. Moreover, the direction of the force $F_x\hat{\imath} + F_y\hat{\jmath}$, where F_x and F_y are given by (10), will be the steepest direction. i.e. the direction the ball will roll down if you gently place it on the terrain.

Now, we introduce gradient. Gradient of a function, say U is defined as follows:

$$\nabla U = \frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k} \tag{14}$$

Using this relation (10) can be expressed as follows:

$$\vec{F} = -\nabla U \tag{15}$$

As follows, it can also be proven that the path independence of the line integral implies that the line integral vanishes for any closed path:

$$\int_A \vec{F} \cdot d\vec{s} = \int_B \vec{F} \cdot d\vec{s} \tag{16}$$

$$\int_A \vec{F} \cdot d\vec{s} + \int_{-B} \vec{F} \cdot d\vec{s} = 0 \tag{17}$$

See Fig. 2. Notice that the path $A + (-B)$ is a closed path. As A and B are any arbitrary paths, we conclude that the line integral along any closed path vanish. In other words, if you start at a point and move around and come back to the original position the altitude difference between before and after the trip will be zero. Force that satisfies this condition is called "conservative force." The condition upon which a force is conservative may seem quite stringent, but all known *fundamental* forces such as gravity, electromagnetic force, strong force and weak force are conservative. Nevertheless, non-conservative forces do exist, if the forces concerned are not fundamental. A good example is a frictional force. In the presence of frictional force, the longer your path, the more you lose energy. Certainly, in such a case, the energy loss is path-dependent.

Problem 1. Why is the contour on maps (i.e. the line that connects all the points with the same height) always perpendicular to the steepest direction? (Hint[1])

Problem 2. Show

$$\nabla\sqrt{x^2+y^2+z^2} = \frac{x\,\hat{i} + y\,\hat{j} + z\,\hat{k}}{\sqrt{x^2+y^2+z^2}} \tag{18}$$

Problem 3. Consider the following line integral:

$$\int_N \vec{F} \cdot d\vec{s} \tag{19}$$

where $\vec{F} = 2y\,\hat{x} + x\,\hat{y}$ and N is given by the straight line from $(0,0)$ to $(1,0)$ plus the straight line from $(1,0)$ to $(0,1)$ plus the straight line from $(0,1)$ to $(0,0)$. (Hint[2]) From the answer to this question, can we determine whether $\vec{F}$ is a conservative force or not? If we can, is it one? In our later article "Curl and Green's theorem" we will teach you a formula that lets you check whether a force is conservative.

Summary

- Energy conservation can be written as $E = K + U$ where

$$K = \frac{1}{2}mv^2, \qquad U(\vec{s}) = -\int F(\vec{s})d\vec{s}$$

[1] Use (12).

[2] You will see that the hardest part of computation is the straight line from $(1,0)$ to $(0,1)$. In this part, express $\vec{F}$ in terms of y and $d\vec{s} = dx\,\hat{x} + dy\,\hat{y}$ in terms of dy using the fact that $x = 1 - y$ which implies $dx = -dy$

K is called kinetic energy and U is called potential energy.

- For the potential energy to be well-defined, the above line integral must be independent of the path taken once their endpoints are fixed.
- Gradient of U is defined by

$$\nabla U = \frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}$$

- $\vec{F} = -\nabla U$. This $\vec{F}$ satisfies the condition that the line integral just mentioned is independent of the path taken.
- Such a force is called "conservative force." All elementary forces are conservative forces.