Kinetic energy, potential energy and angular momentum in polar coordinate and Kepler's second law

In later articles, we will use the polar coordinate to express the equation of motion of planets. There, everything will be derived by placing the origin of the polar coordinate on Sun. This article serves as the first step toward this goal.

Recall that the kinetic energy is given as follows in Cartesian coordinate:

$$\frac{1}{2}mv^2 = \frac{1}{2}m(v_x^2 + v_y^2) \tag{1}$$

where $v_x = dx/dt$ and $v_y = dy/dt$. What would be the analogous equation in polar coordinate? We have:

$$x = r\cos\theta, \quad y = r\sin\theta \tag{2}$$

Differentiating using Leibniz rule, we get:

$$\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\frac{d\theta}{dt}\sin\theta \tag{3}$$

$$\frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\frac{d\theta}{dt}\cos\theta \tag{4}$$

Therefore, we obtain:

$$v^2 = v_x^2 + v_y^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \tag{5}$$

If we use the following notation:

$$\dot{f} = \frac{df}{dt} \tag{6}$$

(5) can be re-expressed as:

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + (r\dot{\theta})^2 \tag{7}$$

We have an obvious interpretation for this formula. $\dot{r}$ is the velocity in the radial direction and $r\dot{\theta}$ is the velocity in the angular direction. (Remember that in our earlier article "centripetal force" we obtained that the velocity in angular direction was given by $r\omega$. In this case, $\dot{\theta}$ is simply ω.) The total velocity squared is the sum of the square of each of them, as they are orthogonal to each other. If you are not clear what I mean, see Fig. 1 in which we call the unit vector (i.e. a vector with magnitude 1) in radial direction $\hat{r}$ and the unit vector in angular direction $\hat{\theta}$. In any case, what we said so far in the beginning of this paragraph can be re-expressed mathematically as:

$$\vec{v} = \frac{d\vec{s}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} \tag{8}$$

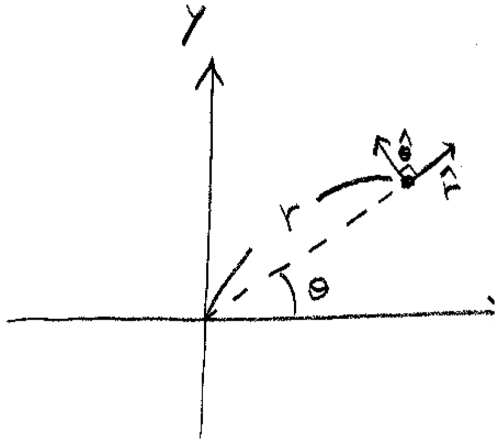

Figure 1: $\hat{r}$ and $\hat{\theta}$ in polar coordinate

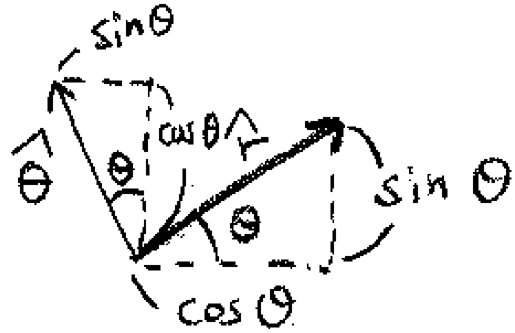

Figure 2: $\hat{r}$ and $\hat{\theta}$ in terms of $\hat{\imath}$ and $\hat{\jmath}$

or, in other words, by canceling out dt in the denominator:

$$d\vec{s} = dx\,\hat{\imath} + dy\,\hat{\jmath} = dr\,\hat{r} + rd\theta\,\hat{\theta} \tag{9}$$

$\hat{r}$ and $\hat{\theta}$ can be actually obtained by plugging (3) and (4) to (8). We get (**Problem 1.**):

$$\hat{r} = \cos\theta\;\hat{\imath} + \sin\theta\;\hat{\jmath} \tag{10}$$

$$\hat{\theta} = -\sin\theta\;\hat{\imath} + \cos\theta\;\hat{\jmath} \tag{11}$$

This can be actually obtained geometrically as well. See Fig. 2. In any case, one can indeed check that they are orthogonal and unit vectors. In other words, using (10) and (11), we obtain

$$\hat{r}\cdot\hat{r} = \hat{\theta}\cdot\hat{\theta} = 1, \qquad \hat{r}\cdot\hat{\theta} = 0 \tag{12}$$

This allows us to re-derive (5) by using (8) as follows:

$$v^2 = \vec{v}\cdot\vec{v} = \left(\frac{dr}{dt}\right)^2 \hat{r}\cdot\hat{r} + 2r\frac{dr}{dt}\frac{d\theta}{dt}\hat{r}\cdot\hat{\theta} + r^2\left(\frac{d\theta}{dt}\right)^2 \hat{\theta}\cdot\hat{\theta} \tag{13}$$

$$= \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \tag{14}$$

In conclusion, the kinetic energy is given as follows:

$$\frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{15}$$

Now to the potential energy. In the generic case, elementary force such as gravitation only depends on the distance. In our case, the gravitational force only depends on the distance to the Sun and is in radial direction. (Precisely speaking, negative $\hat{r}$ direction i.e. toward center.) So, we can express the gravitational force $\vec{F}$ as follows:

$$\vec{F} = F(r)\hat{r} \tag{16}$$

Therefore, using (9), we have

$$\begin{aligned} U(r) &= -\int \vec{F} \cdot \vec{ds} && (17) \\ &= -\int F(r)\hat{r} \cdot (dr\ \hat{r} + rd\theta\ \hat{\theta}) && (18) \\ &= -\int F(r)dr && (19) \end{aligned}$$

In the case of Newtonian gravitation, we have: $F(r) = -GMm/r^2$ where M is the Sun's mass and the negative sign is used to denote the fact that the force is directed toward negative $\hat{r}$ direction (i.e. toward center). By plugging this to the above formula, we can see the following:

$$U(r) = -\frac{GMm}{r} + \text{constant} \tag{20}$$

where the constant term is due to integration constant. However, it is customary to set this constant zero. Therefore, we have $U(r) = -GMm/r$.

Problem 2. Using the chain-rule, show

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} \tag{21}$$

Problem 3. Let's say that you launch a rocket from the ground which is at distance R from the center of the Earth with speed v_0. Let's say that the mass of rocket is m and the mass of the Earth is M. We will further assume that in the universe there are only this rocket and the Earth, ignoring the gravitation of the Sun and other planets. Given this, what is the total energy (i.e. the sum of kinetic energy and the potential energy) of this rocket upon launching?

Problem 4. Assume that the rocket reaches a point infinitely far away from the Earth. What is its potential energy at this point? (Hint:[1]) How about its kinetic energy? (Hint:[2])

Problem 5. Using the result of Problem 4 and the fact that the kinetic energy can never be less than zero, obtain the minimum speed upon launching which the rocket needed to reach a point infinitely far away from the Earth. This is called "escape velocity." In other words, if you shoot a rocket with a speed less than the escape velocity, it will never reach region far enough to be truly free from the gravity of the Earth; at a certain point, it will move toward the Earth.

As an aside, black hole is a region where the escape velocity is greater than the speed of light. According to Einstein's theory of relativity nothing can move faster than speed of light. Therefore, nothing can escape from a black hole. The simplest black hole has a shape of sphere with radius called "Schwarzschild radius" which depends on G, (the Newton's constant) M, (the mass of the black hole) and c (the speed of light.) The Schwarzschild radius can be obtained in a wrong method by using non-relativistic, Newtonian formulas by setting escape

[1] Use $U = -GMm/r$

[2] The sum of the kinetic energy and the potential energy is equal to the one obtained in Problem 3, because of the conservation of the energy.

velocity equal to c. (**Problem 6.** Obtain Schwarzschild radius.) Surprisingly, this yields the correct answer which one can obtain from the exact, fully relativistic treatment which we will show you in our later article "A Relatively Short Introduction to General Relativity.". When I was young, I read about Schwarzschild radius from a cartoon book on relativity and found out that this was exactly equal to the value obtained non-relativistically. Therefore, I thought that it couldn't be the right value and the book had an error. I told so to my physics tutor but she didn't know enough general relativity to answer my question.

Problem 7. First, prove $\dot{\hat{r}} = \dot{\theta}\,\hat{\theta}$ and $\dot{\hat{\theta}} = -\dot{\theta}\,\hat{r}$ using (10) and (11). Using this result, show that the acceleration is given as

$$\vec{a} = \frac{d^2\vec{s}}{dt^2} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} \tag{22}$$

where $\ddot{r} = d^2r/dt^2$ and $\ddot{\theta} = d^2\theta/dt^2$. (Hint[3]). Notice that the force in polar coordinate system is given as follows.

$$\vec{F} = m\vec{a} = (m\ddot{r} - mr\dot{\theta}^2)\hat{r} + (2m\dot{r}\dot{\theta} + mr\ddot{\theta})\hat{\theta} \tag{23}$$

If the force is in radial direction, i.e., $\vec{F} = F_r\hat{r}$, the above equation can be re-expressed as

$$m\ddot{r} = F_r + mr\dot{\theta}^2 \tag{24}$$

$$2m\dot{r}\dot{\theta} + mr\ddot{\theta} = 0 \tag{25}$$

Now, let's look at the first equation. In addition to the radial force F_r, the object sees as if it receives extra force $mr\dot{\theta}^2$ in positive radial direction (i.e., outward force.) This is exactly centrifugal force.

We can also interpret the second equation. Let's multiply by r on the both-hand sides. Then, it is easy to see that the following equation must be satisfied if the second equation is satisfied.

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = 0 \tag{26}$$

Notice now that the above equation can be re-expressed as

$$\frac{d(mr^2\dot{\theta})}{dt} = 0 \tag{27}$$

Therefore, we see that $mr^2\dot{\theta}$ is conserved (i.e., constant) if the force is radial. Now, remember that angular momentum is given by $L = I\omega$ where the moment of inertia is $I = \sum_i m_i r_i^2$. In our case, this is $I = mr^2$. As ω is $\dot{\theta}$, we see that the conserved quantity $mr^2\dot{\theta}$ is precisely angular momentum! We conclude that angular momentum is conserved in central force (i.e. force that are directed in radial direction.) In our later article "Re-visiting angular momentum conservation in central force" we will derive again that the quantity $mr^2\dot{\theta}$ is indeed angular momentum and is conserved from a more mathematical definition of angular momentum.

[3]Differentiate $\vec{v} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}$ with respect to time by using Leibniz rule.

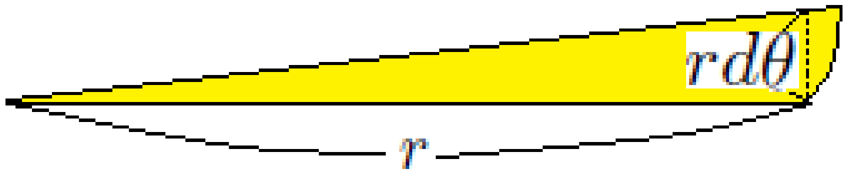

Figure 3: Kepler's second law

Now, we can prove Kepler's second law algebraically. Kepler's second law asserts that the line connecting the Sun and the planet swipes equal amount of area during equal amount of time. Now, let's express this area using r and $\dot{\theta}$. For a very small angle $d\theta$, the area swiped by the line can be shown equal to $\frac{1}{2}r \cdot rd\theta$. See Fig.3. The area concerned is the small triangle whose base is r and whose height is $rd\theta$. Therefore, per unit time, the line swipes the area by following:

$$\frac{\frac{1}{2}r^2 d\theta}{dt} = \frac{1}{2}r^2\dot{\theta} \tag{28}$$

As $mr^2\dot{\theta}$ is a constant, we conclude that the above formula is constant.

Problem 8. We know that $U(r) = -GMm/r$ must be valid even when we consider an object moving in 3 dimensional space instead of a plane in 2d as considered in this article. This is so, because (19) is still valid where $F(r) = -GMm/r^2$ as before. Nevertheless, as an exercise let's check that this potential energy gives out the correct gravitational force in 3d Cartesian coordinate. To this end, remember that the distance to the origin is given by $r = \sqrt{x^2 + y^2 + z^2}$ from Pythagorean theorem. Note also that the vector pointing from origin to the point where object is located is given by $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$. Then, the unit vector pointing in positive radial direction is given by

$$\hat{r} = \frac{\vec{r}}{r} \tag{29}$$

(Check that this is indeed unit vector!) Given this, check $U(r) = -GMm/r$ gives rise to the force $F(r) = (-GMm/r^2)\hat{r}$ as expected. (Hint[4])

Summary

- The velocity vector can be expressed in the 2-d Cartesian coordinate and the polar coordinate as

$$\vec{v} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}$$

- $\hat{r} \cdot \hat{r} = \hat{\theta} \cdot \hat{\theta} = 1, \qquad \hat{r} \cdot \hat{\theta} = 0$

[4]Use $\nabla U = \dfrac{\partial U}{\partial r}\nabla r$ (i.e. the chain rule) and the result of Problem 2 of "Kinetic energy and Potential energy in three dimensions, Line integrals and Gradient"

- Kepler's second law is the consequence of angular momentum conservation.
- The gravitational potential is given by

$$U(r) = -\frac{GMm}{r}$$

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- $\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta}$
- $\hat{r} = \frac{\vec{r}}{r}$