Kaluza-Klein theory

A very well-known scenario for extra dimension is Kaluza-Klein theory. In 1919, to Einstein, Kaluza suggested a metric for his five-dimensional theory, and showed that this metric leads to the unification of gravity and Maxwell's electromagnetic field. His work was published in 1921. Let us review how this works.

Imagine that we are in five dimensions, with metric components $g^{(5)}_{MN}$, $M, N = 0, 1, 2, 3, 4$ and that the spacetime is actually of topology $\mathbb{R}^4 \times S^1$, and so has one compact direction (S^1 denotes a circle). So we will have the usual four dimensional coordinates on $\mathbb{R}^4$, x^μ ($\mu, \nu = 0, 1, 2, 3$) and a periodic coordinate:

$$x^4 \sim x^4 + 2\pi R \tag{1}$$

where $2\pi R$ is the size of extra dimension. In other words, x^4 and $x^4 + 2\pi R$ denote the same point.

Now, under the five-dimensional coordinate transformation $x'^M = x^M + \epsilon^M(x)$, the five-dimensional metric transforms as follows:

$$g^{(5)'}_{MN} = g^{(5)}_{MN} - \partial_M \epsilon_N - \partial_N \epsilon_M \tag{2}$$

Given this, let us assume that the metric doesn't depend on the periodic coordinate, x^4. Then, we immediately see the followings:

$$\epsilon^\nu = \epsilon^\nu(x^\mu) \tag{3}$$

$$\epsilon^4 = \epsilon^4(x^\mu) \tag{4}$$

which means,

$$x^{\mu'} = \psi^\mu(x^0, x^1, x^2, x^3) \tag{5}$$

$$x^{4'} = x^4 + \epsilon^4(x^0, x^1, x^2, x^3) \tag{6}$$

They have obvious physical interpretations. The first one is the usual four-dimensional diffeomorphism invariance. The second one is an x^μ-dependant isometry(rotation) of the circle; one has a complete freedom of choosing which point on the circle is $x^4 = 0$. In other words, this choice does not matter at all.

Then, from (2), $g^{(5)}_{44}$ is invariant, as ϵ_4 does not depend on x^4. On the other hand, from (2) and (6), we have:

$$g^{(5)'}_{\mu 4} = g^{(5)}_{\mu 4} - \partial_\mu \epsilon_4 \tag{7}$$

However, from the four dimensional point of view, $g^{(5)}_{\mu 4}$ is a vector, having one 4-dimensional index (i.e., μ). This vector is proportional to what we will call A_μ. Thus, the above equation is simply a $U(1)$ gauge transformation for the electromagnetic potential: $A'_\mu = A_\mu - \partial_\mu \Lambda$ if we call the proportionality constant, k, and make the following identification:

$$g^{(5)}_{\mu 4} = kA_\mu, \qquad \epsilon_4 = k\Lambda \tag{8}$$

So the $U(1)$ of electromagnetism can be thought of as resulting from compactifying gravity, the gauge field being an internal component (i.e., $\mu 4$) of metric. We also see that, in this picture, $U(1)$ gauge freedom comes from (6), our freedom to choose which point on the circle is $x^4 = 0$.

Problem 1. Check that, under (6), we have

$$dx^{4'} + kA'_\mu dx^\mu = dx^4 + kA_\mu dx^\mu \tag{9}$$

In other words, we see that $dx^4 + kA_\mu dx^\mu$ is gauge invariant.

Assuming $g^{(5)}_{44} = 1$ from its invariance under gauge transformation and by rescaling, these considerations lead to the following metric:

$$ds^2 = g^{(5)}_{MN} dx^M dx^N = g^{(4)}_{\mu\nu} dx^\mu dx^\nu + (dx^4 + kA_\mu dx^\mu)^2 \tag{10}$$

as only the gauge invariant form must appear in the metric. From this expression, we have

$$g^{(5)}_{\mu\nu} = g^{(4)}_{\mu\nu} + k^2 A_\mu A_\nu, \qquad g^{(5)}_{\mu 4} = g^{(5)}_{4\mu} = kA_\mu, \qquad g^{(5)}_{44} = 1 \tag{11}$$

Now, one can easily check that the inverse metric is given as follows:

$$g^{\mu\nu}_{(5)} = g^{\mu\nu}_{(4)}, \qquad g^{\mu 4}_{(5)} = g^{4\mu}_{(5)} = -kA^\mu, \qquad g^{44}_{(5)} = 1 + k^2 A_\alpha A^\alpha \tag{12}$$

(**Problem 2.** Check the above equation by multiplying it by (11) to obtain the identity matrix.)

Given this, the five-dimensional Ricci scalar can be re-expressed as the four-dimensional one and the electromagnetic field tensor as follows:

$$R^{(5)} = R^{(4)} - \frac{1}{4} k^2 F_{\mu\nu} F^{\mu\nu} \tag{13}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as usual. One can also check that the determinant of $g^{(5)}_{MN}$ is same as the one of $g^{(4)}_{\mu\nu}$. We can roughly see this as follows, even though this is not rigorous.

$$\det \begin{pmatrix} 1 & kA_\mu \\ kA_\nu & g^{(4)}_{\mu\nu} + k^2 A_\mu A_\nu \end{pmatrix} = 1 \times \left(g^{(4)}_{\mu\nu} + k^2 A_\mu A_\nu \right) - kA_\mu \times kA_\nu = g^{(4)}_{\mu\nu} \tag{14}$$

Now, if we denote $G^N_{(4)}$ as 4-dimensional Newton's constant and $G^N_{(5)}$ as its 5-dimensional counterpart, the Einstein-Hilbert action in 5d becomes:

$$S = \frac{1}{16\pi G^N_{(5)}} \int d^5x(-g_{(5)})^{1/2} R^{(5)} \tag{15}$$

$$= \frac{2\pi R}{16\pi G^N_{(5)}} \int d^4x(-g_{(4)})^{1/2}(R^{(4)} - \frac{1}{4}k^2 F_{\mu\nu}F^{\mu\nu}) \tag{16}$$

Therefore, up to some normalization factors, Einstein-Hilbert action in five-dimensional Kaluza-Klein theory reproduces Einstein-Hilbert action in four-dimensional theory and Maxwell-Lagrangian, which means the unification of gravity and electromagnetism. Now, let's determine the normalization factors. The above action must be equal to

$$S = \int d^4x(-g_{(4)})^{1/2}(\frac{1}{16\pi G^N_{(4)}} R^{(4)} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}) \tag{17}$$

Therefore, we conclude

$$\frac{2\pi R}{G^N_{(5)}} = \frac{1}{G^N_{(4)}}, \qquad k = \sqrt{16\pi G^N_{(4)}} \tag{18}$$

where $2\pi R$ is the size of extra-dimension, as stated before.

Now comes Klein's work. After quantum mechanics was formulated, Klein showed in 1926 that one could determine the size of extra-dimension in Kaluza's scenario from quantum mechanics. This is something that we will show in the rest of the article.

To this end, we need to first find an explicit geodesic equation. As was the case in our earlier article "An Introduction to General Relativity," it turns out that extremizing the square of the line element instead of line element is more convenient. After all, the former is equivalent to the latter, when we parametrize the path by the proper time. If we denote the proper time by τ, and $g^4_{\mu\nu}$ by $g_{\mu\nu}$, what we want to extremize is the following:

$$L = \frac{1}{2}m\left(\left(\frac{dx^4}{d\tau} + kA_\mu \frac{dx^\mu}{d\tau}\right)^2 + g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) \tag{19}$$

where m is the mass of the particle concerned and the overall factor $\frac{1}{2}m$ is for future convenience. Now, let's obtain the "momentum" and its equation of motion for this "Lagrangian." We have:

$$p_4 = \frac{\partial L}{\partial(dx^4/d\tau)} = m\left(\frac{dx^4}{d\tau} + kA_\mu\frac{dx^\mu}{d\tau}\right) \tag{20}$$

$$\frac{dp_4}{d\tau} = \frac{\partial L}{\partial x^4} = 0 \tag{21}$$

$$p_\mu = \frac{\partial L}{\partial(dx^\mu/d\tau)} = mg_{\mu\nu}\frac{dx^\nu}{d\tau} + kp_4 A_\mu \tag{22}$$

$$\frac{dp_\mu}{d\tau} = \frac{\partial L}{\partial x^\mu} = \frac{1}{2}m\partial_\mu g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + kp_4\frac{\partial A_\nu}{\partial x^\mu}\frac{dx^\nu}{d\tau} \tag{23}$$

(21) shows that p_4 is a conserved quantity. Also, if you remember the following formula from "Electrodynamics in the Lagrangian and the Hamiltonian formulations,"

$$p_x = m\ddot{x} + qA_x, \qquad p_x = m\ddot{y} + qA_y, \qquad p_z = m\ddot{z} + qA_z \tag{24}$$

(22) turns out to be the momentum, provided

$$p_4 = \frac{q}{k} \tag{25}$$

Therefore, we indeed see that the charge conservation implies that p_4 is a conserved quantity. One can also check that this choice of p_4 yields the correct equation of motion in the presence of electromagnetic field. Plugging (22) to (23) yields:

$$\frac{dp_\mu}{d\tau} = m\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) + kp_4\frac{\partial A_\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau} \tag{26}$$

Equating this with the right-hand side of (23) yields the following (**Problem 3.** Hint[1]):

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = kp_4F_{\nu\mu}\frac{dx^\nu}{d\tau} \tag{27}$$

With the identification of $q = kp_4$, this equation exactly becomes the equation of motion in the presence of electromagnetic field $F_{\nu\mu}$. (**Problem 4.** Check that the above equation reduces to the following in the flat space:

$$m\ddot{\vec{r}} = q\vec{E} + q\dot{\vec{r}} \times \vec{B} \tag{28}$$

where $\vec{r} = (x, y, z)$.)

Given all these, let's obtain the size of the extra dimension. The wave function of a particle with momentum $\vec{p}$ is given as follows

$$\psi(x) = Ae^{i\vec{p}\cdot\vec{x}/\hbar} = e^{i(p_1x^1+p_2x^2+p_3x^3)/\hbar}e^{ip_4x^4/\hbar} \tag{29}$$

Focusing on the last factor of the above equation and using (1), we must have:

$$e^{ip_4x^4/\hbar} = e^{ip_4(x^4+2\pi R)/\hbar} \tag{30}$$

which implies:

$$p_4(2\pi R)/\hbar = 2\pi N, \qquad \rightarrow \qquad p_4 = N\frac{h}{2\pi R} \tag{31}$$

for an integer N. Using $q = kp_4$, we have:

$$q = N\frac{kh}{2\pi R} \tag{32}$$

So, we derived the fact that an electric charge must be the integer multiples of the fundamental charge $h/(2\pi Rk)$. According to quantum chromodynamics, the charge of quark is $\pm e/3$ or $\pm 2e/3$ where $-e$ is the charge of the electron. Therefore, the fundamental charge seems to be $e/3$. Therefore, we obtain:

$$2\pi R = \frac{3h\sqrt{16\pi G^N_{(4)}}}{e} \approx 2.5 \times 10^{-32}\text{meter} \tag{33}$$

[1] See Section 12 of "An Introduction to General Relativity."

In terms of Planck length l_p and the fine structure constant α defined as follows,

$$l_p = \sqrt{\frac{\hbar G}{c^3}}, \qquad \alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137.03\cdots} \tag{34}$$

we have:

$$2\pi R = \frac{3(4\pi)^{3/2}}{\sqrt{\alpha}} l_p \tag{35}$$

In my paper with Brian Kong "Black hole entropy predictions without Immirzi parameter and Hawking radiation of single-partition black hole," I suggested a way to calculate the size of the extra dimension. Since it can be expressed in terms of fine structure constant, if somebody succeeds in obtaining it using my suggestion, we will find a way to calculate the fine structure constant.

Summary

- Let's say that the spacetime has a topology of $\mathbb{R}^4 \times S^1$, and the metric doesn't depend on S^1, the extra dimension part. Then, the theory of general relativity in such a case becomes the theory of general relativity in 4d and Maxwell theory in 4d.

- In particular, the following symmetry

  $$x^{4'} = x^4 + \epsilon^4(x^0, x^1, x^2, x^3)$$

  gives the gauge symmetry, and the conservation of momentum along the extra dimension gives the conservation of electric charge.