Newton's proof of Kepler's second law

In this article, we will present Newton's proof of Kepler's second law as promised in an earlier article. Let me state again Kepler's second law. During equal amount of time, the straight line connecting the Sun and the planet swipes equal amount of area. See Fig.1. If it equally takes t seconds for the planet to travel each interval, the area swiped by the planet is equally A.

Before proving Kepler's second law, we will first prove it in the special case when there is no gravitational force. See Fig.2. The Sun is located at O, and the planet moves from A to B in t seconds, B to C in t second, C to D in t seconds. Since there is no gravitational force, each interval has the same length. As the three triangles ΔOAB, ΔOBC, ΔOCD have the same length of base and the same height, their areas are the same.

Now, let's apply gravitational force. See Fig.3. If there were no gravitational force, the planet will reach C, t seconds after it reaches B. However, since there is a gravitational force, the planet reaches C', pulled toward the Sun. Since the line segments $\overline{OB}$ and $\overline{CC'}$ are parallel, the area of the triangles ΔOBC and $\Delta OBC'$ are the same. ($\overline{CC'}$ may not seem to be directed toward the Sun, if $\overline{OB}$ and $\overline{CC'}$ are parallel, but in the limit in which the interval t goes to zero, $\overline{CC'}$ is exactly directed toward the Sun, as the direction toward the Sun at point B is exactly the same as the direction toward the Sun at point C, because the location B and the location C become infinitely close.) As the areas of ΔOAB and ΔOBC are the same, we conclude the area of ΔOAB and $\Delta OBC'$ are the same. Now, we can repeat this process again. We know that the areas of $\Delta OBC'$ and $\Delta OC'D'$ are the same, as the

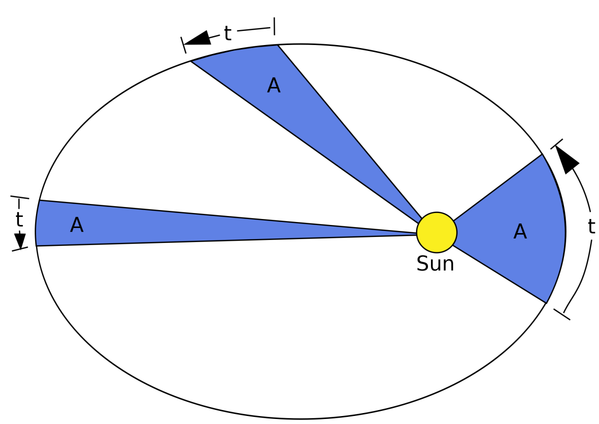

Figure 1: Kepler's second law

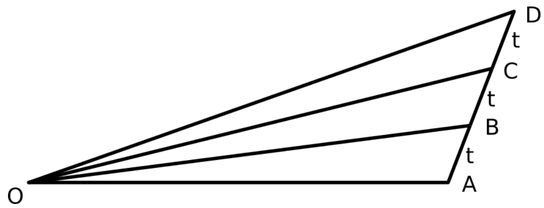

Figure 2: Without a gravitational force

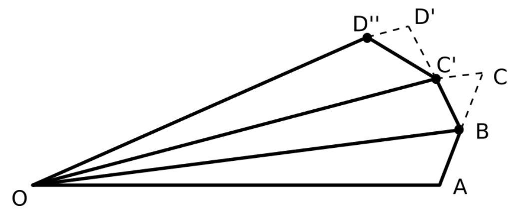

Figure 3: Under a gravitational force

lengths of $\overline{BC'}$ and $\overline{C'D'}$ are the same. We know that the areas of $\Delta OC'D$” and $\Delta OC'D''$ are the same as $\overline{D'D''}$ is parallel to $\overline{OC'}$. Thus, the area of $\Delta OBC'$ is equal to the one of $\Delta OC'D''$. In conclusion, the areas of ΔOAB, $\Delta OBC'$, $\Delta OC'D''$ are all the same. Thus, we proved Kepler's second law.

Summary

- Kepler's second law says that during equal time the line connecting the planet and the Sun swipes the equal amount of area.
- Kepler's second law can be derived from the fact that the direction of the Sun's gravitational attraction of the planet is toward the Sun.

(Fig.1 is from `https://commons.wikimedia.org/wiki/File:Kepler2.gif`)