Kronecker delta symbol

Kronecker delta symbol is defined by following rules:

$$\delta_{ab} = 1 \quad \text{if} \quad a = b \tag{1}$$

$$\delta_{ab} = 0 \quad \text{if} \quad a \neq b \tag{2}$$

For example, $\delta_{12} = 0$, $\delta_{33} = 1$ and so on.

Sometimes, it's more useful to write the Kronecker delta symbol with one upper index and one lower index as following rather than two lower indices as above.

$$\delta^a_b = 1 \quad \text{if} \quad a = b \tag{3}$$

$$\delta^a_b = 0 \quad \text{if} \quad a \neq b \tag{4}$$

If we use Kronecker delta symbol, we can express the identity matrix I as follows.

$$(I)_{ab} = \delta_{ab} \tag{5}$$

where $(I)_{ab}$ denotes (a, b)th components of the matrix I.

Furthermore, it is easy to check that

$$\sum_b (A)_{ab}\delta_{bc} = (A)_{ac} \tag{6}$$

This is so, because δ_{bc} is zero, unless $b = c$. So, the only contribution of the sum comes from the case when $b = c$. The above relation can be expressed in the language of matrix as follows:

$$AI = A \tag{7}$$

Similarly, $IA = A$ can be expressed as

$$\sum_b \delta_{ab}(A)_{bc} = (A)_{ac} \tag{8}$$

Using the Einstein summation convetion, it can be re-expressed as

$$\delta^a_b A^b_c = A^a_c \tag{9}$$

Here, you see that the dummy variable b in A^b_c is replaced by a, due to the Kronecker delta symbol δ^a_b.

Problem 1. In 4-dimension, if the Einstein-summation convention is assumed, what is δ^a_a? How about $\delta^a_b\delta^b_a$? How about $\delta^a_a\delta^b_b$?

Problem 2. Simplify the following expressions.

$$\delta^a_b A_a =?, \qquad \delta^c_d B^d_c =?$$

$$\delta^a_b A^{bc} =?, \qquad \delta^a_b \delta^d_c A^{bc} =?, \qquad (\delta^a_b\delta^d_c - \delta^a_c\delta^d_b)A^{bc} =?$$

Summary

- Kronecker delta symbol is defined by

$$\begin{aligned} \delta_{ab} &= 1 \quad \text{if} \quad a = b \\ \delta_{ab} &= 0 \quad \text{if} \quad a \neq b \end{aligned}$$

- Kronecker delta symbol is the components of the identity matrix.